Worksheet 8.2,4

Max's Lecture
MATH 55

July 25, 2019

Exercise A (example in book). Verify that $H_n = -1$ is a solution to the recurrence $H_n = 2H_{n-1} + 1$. Use this particular solution to find all solutions for the recurrence. Then, use the initial value $H_1 = 1$ to find a closed form solution to the towers of Hanoi problem.

In previous worksheet.

Exercise B. You are making waffles, and are topping them with scoops of whipped cream, spoonfuls of sprinkles, and gummy worms. A proper waffle must satisfy the following:

- have at least 3 scoops of whipped cream
- have at most 1 spoonful of sprinkles
- have an even number of gummy worms

How many different kinds of proper waffles can I make with n total toppings?

This is an application of convolution rule.

Let $W(x)$ be # ways to make a "proper" waffle only considering whipped cream (and only considering the restrictions on whipped cream).

$$W(x) = x^3 + x^4 + \dots = \frac{x^3}{1-x}$$

Let $S(x)$ be the same idea, but with sprinkles,

$$S(x) = 1 + x$$

Let $G(x)$ be the same idea, but with gummy worms

$$G(x) = 1 + x^2 + x^4 + \dots = \frac{1}{1-x^2}$$

Let $P(x)$ be generating function for proper waffles (with all ingredients and restrictions).

By the convolution rule:

$$P(x) = W(x) \cdot S(x) \cdot G(x) = \frac{x^3}{1-x} \cdot \frac{1}{1-x^2} \cdot (1+x) =$$

$$\frac{x^3}{(1-x)^2} = \underbrace{x^3 \sum_{n=0}^{\infty} (n+1)x^n}_{\text{The fact that this is equal to } \frac{1}{(1-x)^2} \text{ was shown in class.}} = \sum_{n=0}^{\infty} (n+1) x^{n+3}$$

So you can make $n-2$ waffles with a total of n toppings!

Exercise C. Give an expression for the generating function of the number of ways to distribute n identical toys to 3 children such that each child gets at least 2 toys.

Again, we write a separate generating function for each child.

Child 1: $0 + 0x + x^2 + x^3 + \dots = \frac{x^2}{1-x}$

There is one way to give the child n toys if $n \geq 2$, 0 ways if $n = 0$ or 1.

Child 2: $x^2 + x^3 + \dots = \frac{x^2}{1-x}$

Child 3: $x^2 + x^3 + \dots = \frac{x^2}{1-x}$.

So by convolution, the generating function for # ways to distribute the toys between the children is

$\left(\frac{x^2}{1-x}\right)^3$.