Problems

1. Let $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$ be a lin. homog. recurrence with constant coefficients.
Show that if $\{s_n\}$ and $\{t_n\}$ both satisfy the recurrence, so does $b_1 s_n + b_2 t_n$ for all $b_1, b_2 \in \mathbb{R}$.

2. Verify that if r_1, r_2 are distinct solutions to the ~~recurrence~~ ~~notation~~ char eq for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then

$\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the recurrence for any choice of $\alpha_1, \alpha_2 \in \mathbb{R}$.

3. Find all solutions to the "Fibbonacci recurrence",
$a_n = a_{n-1} + a_{n-2}$.

4. Recall that the in. conditions of the fibbonacci sequence are $f_0 = 0, f_1 = 1$. Find the closed form of the fibbonacci sequence.

5. Find all solutions of the following recurrence:

   A. $a_n = 6a_{n-1} - 9a_{n-2}$

   B. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$.

6. Show that $\{a_n\} = -1$ is a solution to the recurrence $a_n = 2a_{n-1} + 1$.
Now use Thm 5 to find all solutions to $H_n = 2H_{n-1} + 1$.

Solutions

$\boxed{1}$ We ~~substitute~~ assume $\{s_n\}$, $\{t_n\}$ satisfy the recurrence

We substitute $b_1 s_n + b_2 t_n$ into the right hand side of the recurrence:

$$c_1(b_1 s_{n-1} + b_2 t_{n-1}) + \dots + c_k(b_1 s_{n-k} + b_2 t_{n-k}) =$$

$$b_1\left[c_1 s_{n-1} + \dots + c_k s_{n-k}\right] + b_2\left[c_1 t_{n-1} + \dots + c_k t_{n-k}\right]$$

$= b_1 s_n + b_2 t_n$, which shows that $b_1 s_n + b_2 t_n$ does indeed satisfy the recurrence.

$\boxed{2}$ Since r_1, r_2 are solutions to the char. equation,

$$☆ \quad r_1^2 - c_1 r_1 - c_2 = 0 \quad \text{and} \quad r_2^2 - c_2 r_2 - c_2 = 0$$

We now verify that $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfy the recurrence.

$$c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + r_2^{n-2}) =$$

$$\alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$$

$$= \alpha_1 r_1^{n-2}(r_1^2) + \alpha_2 r_2^{n-2} r_2^2 = \alpha_1 r_1^n + \alpha_2 r_2^n.$$

since by rearranging ☆ we get $r_1^2 = c_1 r_1 + c_2$.

$\boxed{3}$ The char eq is $r^2 - r - 1 = 0$, which has roots $\frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$.

So all solutions are of the form

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$\boxed{4.}$ We now use the initial conditions to find α_1, α_2

Since $f_0 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha_1 + \alpha_2 = 0$

and

$f_2 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$,

we solve this system for α_1 and α_2 to get

$$\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}.$$

So $f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$

$\boxed{5}$ $\boxed{A}$ The char eq. is $r^2 - 6r + 9 = 0$

The roots are 3 with multiplicity 2. So the solutions are of the form

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n.$$

$\boxed{B}$ The char eq. is $r^3 + 3r^2 + 3r + 1 = (r+1)^3$.

Solutions are of the form

$$\alpha_1 (-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2 (-1)^n$$

$\boxed{6}$ We verify that $a_n = -1$ is a solution.

$2(-1)+1 = -2+1 = -1$. So $a_n = -1$ satisfies the recurrence.

To describe all solutions, we just need to add solutions to the homogenous equation $H_n = 2H_{n-1}$.
This has char eq $r-2=0$. So homogenous solutions are of the form $H_n = \alpha 2^n$.

So the solutions of the original recurrence are

$H_n = \alpha 2^n - 1$.