Worksheet 7.2 and 7.3

Max's Lecture
MATH 55

July 19, 2019

Exercise A: Example 2 in 7.~~3~~ 2. Suppose that a die is biased so that 3 appears twice as ofthen as each other number but all other options are equally likely. What is the probability that an odd number appears when we roll the dice?

From the given info, we know that $p(1) = p(2) = p(4) = p(5) = p(6)$

Call this quantity p. Furthermore, $p(3) = 2p$.

By the definition of a probability distribution,

$$p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = p + p + p + 2p + p + p = 7p = 1$$

So $p = \frac{1}{7}$. So the prob. dist is given by:

$$p(1) = p(2) = p(4) = p(5) = p(6) = \frac{1}{7}, \quad \text{and } p(3) = \frac{2}{7}$$

The probability that an odd number appears is

$$p(1) + p(3) + p(5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \boxed{\frac{4}{7}}$$

Exercise S. uppose we flip a coin three times, and all 8 possibilities are equally likely. Suppose we know that the event F, that the first coin comes up tails, occurs. Given this information, what is the probability of the event E, that an odd number of tails appears?

From our discussion in class, we know

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

$$p(E \cap F) = \frac{\text{\# outcomes in which first coin is tails and there is an odd \# of tails}}{8} =$$

$\frac{2}{8} = \frac{1}{4}$, since this only occurs for TTT and THH.

$$p(F) = \frac{\text{\# outcomes in which first coin is tails}}{8} = \frac{4}{8} = \frac{1}{2}.$$

So our conditional probability is $p(E|F) = \frac{1/4}{1/2} = \frac{1}{2}$.

Exercise C (example in book). Assume that each of the four ways a family can have 2 children is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent?

No! First, we can kind of guess that they are not independent, since the two events seem to depend on each other. To show this rigorously, we use the fact that E, F are independent iff $p(E \cap F) = p(E)p(F)$.

We compute all 3 quantities:

$$p(E) = \frac{1}{4}, \quad p(F) = \frac{3}{4}, \quad p(E \cap F) = \frac{1}{4}.$$

Since $p(E) \cdot p(F) \neq p(E \cap F)$, these events are not independent.

Exercise D (example in book). Suppose 1 person in 100,000 has a particular rare disease for which their is a fairly accurate diagnostic test. This test is correct 99 percent of the time when given to a person who has the disease, and it is correct 99.5 percent of the time when given to a person at random who does not have the disease.

Given this information, can we find:

1. the probability that a person who tests positive for the disease has the disease?
2. the probability that a person who tests negative for the disease really does not have the disease?

This will be covered on the next worksheet!

(Don't want to spoil it now).

Exercise E.

Prove that the binomial distribution for a given n,p defined in class actually is a probability distribution.

Let n, p be fixed. Recall that our sample space is $\{0, 1, 2, \ldots, n\}$ and that for $0 \le k \le n$,

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

We need to show 2 things:

(i) $0 \le p(k) \le 1$ for all $0 \le k \le n$.

(ii) $\sum_{k=0}^{n} p(k) = 1$

We actually prove (ii) first.

$$\sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \underset{\substack{\uparrow \\ \text{by binomial thm}}}{=} (p + (1-p))^n = 1^n = 1.$$

We can now prove (i):

The formula for $p(k)$ does give a nonnegative number so $p(k) \ge 0$.

Furthermore, since $\sum_{k=0}^{n} p(k) = 1$ and all are nonnegative, $p(k) \le 1$ for each k, as desired.