Worksheet 6.4-5 Solutions.

(Problem Statements will be in black, Solutions will be in Green)

[A] Prove the following 2 identities:

1) Vandermonde's identity: Let $m, n, r \in \mathbb{Z}_{\geq 0}$ with r not exceeding m or n. Then:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

There are several ways to do this. One way is to use the formulas for binomial coeffitions. A combinatorial proof is given in the book on page 442.

2) Hockeystick Identity:

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$ where n and r are positive integers. Question: Why do you think this is called the hockey stick identity?

Note: It is possible to prove this using Pascal's identity, but here we will give a combinatorial proof.

This is a tricky one, so don't panic if it seems difficult.

We give two ways to count the following: The # of ways to use r 0's and $n+1$ 1's to make a bit string

Way 1: There are $r+n+1$ total slots, and we choose which ones are 0's. So we choose r out of $r+n+1$ total, getting $\binom{n+r+1}{r}$.

Way 2: Suppose the $(j+1)$th term is the last term equal to 1. Thus, $n \leq j \leq n+r$. We know there are n 1's and $j-n$ 0's in the first j terms. The last $n+r+1-j$ terms are all determined.

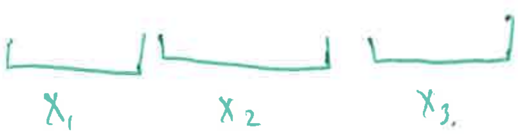
So there are $\binom{j}{j-n}$ ways to make such a string (by choosing where the 0's in the first j terms are. Summing over all possible j's, we get $\sum_{j=n}^{n+r}\binom{j}{j-n}$ Substituting $k=j-n$ (or alternately, $j=n+k$),

We get $\sum_{k=0}^{r}\binom{k+n}{k}$.

So $$\sum_{k=0}^{r}\binom{k+n}{k}=\binom{n+r+1}{r}$$ as desired.

$\boxed{B.}$ This is exercise A on the next worksheet.

$\boxed{C.}$ How many solutions does $x_1+x_2+x_3=11$ have, where x_1, x_2, x_3 are nonnegative integers?



Think of this as putting 11 indistinguishable balls in these 3 distinguishable boxes. By the previous problem, B.4, we know this is $\binom{3+11-1}{11}=\binom{13}{11}$.

What if you know $x_1 \geq 1$, $x_2 \geq 2$?

Then, x_1 has one ball already in it, x_2 has 2 balls already in it. So we have $11-3=8$ balls left to distribute.

So there are $\binom{3+8-1}{8}=\binom{10}{8}$ ways to do this