Worksheet 6.3 and 6.4

Max's Lecture
MATH 55

July 16, 2019

Exercise A (Examples from book). Express each quantity as either a combination or a permutation.

1. The number of ways to select a first prize winner, a second prize winner, and a third prize winner from 100 people at a contest
2. The number of ways to choose which six astronauts to go to mars out of 30 trained people.
3. The number of ways to choose 3 letters from English alphabet. (there are 26 total letters)
4. The number of ways to choose one shirt for me and one for my friend, from my collection of 8 shirts.
5. The number of bit strings of length 8 that contain exactly 3 ones.

1. $P(100,3)$

2. $C(30,6)$

3. $C(26,3)$

4. $P(8,2)$

5. $C(8,3)$ This one is a bit tricky, what you do is choose which 3 indices out of the 8 total you want to be 1. (the rest are automatically 0.)

Exercise B(6.3.21). How many permutations of the letters ABCDEFGH contain:

1. The string ED?
2. The strings BA and GF
3. The strings ACD and CDE?
4. The strings CBA and BED?

1. $\boxed{7!}$ (We treat ED as one letter, so we have 7 "letters" to permute)

2. We treat BA, GF each as a letter. So we have 6 "letters": BA, C, D, E, GF, H. So there are $\boxed{6!}$ ways to permute.

3. In order for ACD and CDE to both be in the string, ACDE has to be in the string. So we have 5 "letters": ACDE, B, F, G, H. So there are $\boxed{5!}$ ways to permute.

4. $\boxed{0}$ ways! It is impossible for CBA and BED to exist in the same string of letters.

Exercise C (adaoted from 6.3.26. How many ways are there for three penguins and six puffins to stand in line so that:

1. All the puffins stand together
2. All the penguins stand together
3. No two penguins stand next to each other.

Note: We are treating the pengs, puffs as distinguishable

Note: There are many equivalent formulations of the following answers.

1. We first arrange the penguins. There are 3! ways to do this

_ | _ | _ | _

penguins.

Now, there are 4 slots to place the block of 6 puffins. Once we make this choice, there are 6! ways to arrange the puffins among themselves.

So in total there are $\boxed{4 \cdot 3! \cdot 6!}$ ways to do this.

2. Very similar reasoning, but this time we get $\boxed{7 \cdot 3! \cdot 6!}$ ways

let me know if you have any questions!

3. We first arrange the puffins, there are 6! ways to do this.

_ | _ | _ | _ | _ | _ | _

puffins

There are 7 slots to place the penguins in, and each must go in its own slot! There are $\binom{7}{3}$ ways to figure out which slots contain penguins.

Then there are 3! ways to arrange the penguins in these 3 slots.

So our total is: $\boxed{\binom{7}{3} 3! \, 6!}$

Exercise D (Example from book). 1. What is the coeffictint of $x^{12}y^{13}$ in $(x+y)^{25}$

2. What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$

1. By binomial theorem it is $\binom{25}{12}$

(or $\binom{25}{13}$, its the same number)

2. The $x^{12}y^{13}$ term is $\binom{25}{12}(2x)^{12}(-3y)^{13}$,

so the coefficient is

$$2^{12}(-3)^{13}\binom{25}{12}.$$

Exercise E(example from book). Prove the identity:

Let n be a positive integer. Then:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

For an extra challenge, try to think of a combinatorial proof!

Let $x = 1$, $y = -1$. Then by the binomial theorem,

$$0 = (1-1)^k = (1+(-1))^k = \sum_{k=0}^{n}\binom{n}{k}1^{(n-k)}(-1)^k =$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k \quad \text{as desired.}$$

Hint for challenge: This can be rephrased as:

# of subsets with even size = # subsets with odd size.

Find a bijection between these two collections.