Math N55– Practice Final
Discrete Mathematics

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Name: ______________________________
Student Number: ________________________

This exam contains 9 pages (including this cover page) and 7 questions. Total of points is 50.
Good luck !

Distribution of Marks

Question	Points	Score
1	9	
2	9	
3	10	
4	6	
5	6	
6	5	
7	5	
Total:	50	

1. If possible for each of the following give an example. If none exist, explain why.

   (a) (3 points) A connected graph with 9 vertices and 7 edges.

   Impossible. A connected graph needs a spanning tree. A tree on 9 vertices has 8 edges. A graph cannot have less edges than it's spanning tree.

   (b) (3 points) A bipartite graph with 15 edges.

   $K_{5,3}$ is an example.

   (c) (3 points) A simple graph that has chromatic number greater than 4.

   K_5 is an example, since K_5 has chromatic number 5

2. Give the cardinality of the following sets. If the cardinality is infinite specify whether countably or uncountably infinite. Justify your answers.

(a) (3 points) The set of reflexive relations on a set of n elements

We know all relations of the form (a,a) are in R since R is reflexive
We just have to determine for each (a,b) where $a \neq b$ whether its in R.
There are $n(n-1)$ such pairs. For each pair we have choices:
include it in R or not.
So there are $2^{n(n-1)}$ such relations

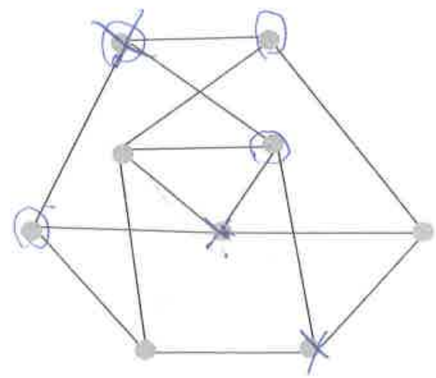
(b) (3 points) The set of regions in a planar graph with 4 vertices each of degree two.

By handshaking lemma, there are $\frac{4 \cdot 2}{2} = 4$ edges
By Euler's formula, there are $v - e + 2 = 4 - 4 + 2 = 2$ regions

(c) (3 points) The set of solutions to the recurrence $a_n = 2a_{n-1} + 6a_{n-2}$

The set of solutions to a lin, ec, homog recurrence
always includes a parameter that can be chosen from
the real numbers.
Thus, since there are uncountably infinite
choices for this parameter, there are
uncountably infinite solutions

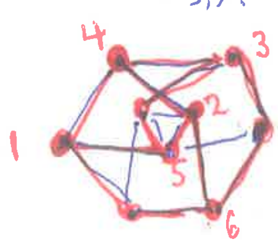
3. Let G be the graph shown below:



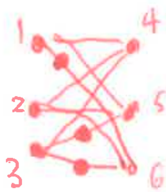
(a) (4 points) Show that G does not have a subgraph homeomorphic to K_5. Hint: Consider the vertex degrees.

Any subgraph homeomorphic to K_5 needs ~~a vertex with degree ≥5. The original graph has no vertex of degree ≥5, so a subgraph also won't have such a vertex~~ at least 5 vertices of degree 4. Since G has only 3 vertices of degree ≥ 4, this is impossible.

(b) (3 points) Show that G is not planar. We find a subgraph homeomorphic to $K_{3,3}$. Consider the following subgraph: (in red)



We redraw this as

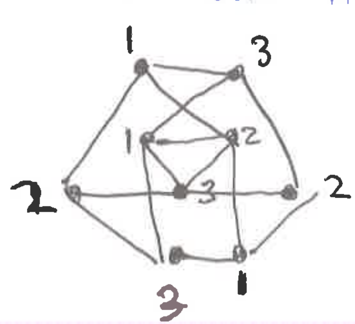


which is homeomorphic to $K_{3,3}$.

(c) (3 points) What is the chromatic number of G? Justify your answer.

It is 3. First, we know we cannot color it in less than 3 colors because it contains a copy of K_3 as a subgraph, which is not 2-colorable.

We can color it with 3 colors, as demonstrated below.



4. (6 points) Prove that if S is a 15 element subset of $\{1, 2, \ldots, 50\}$, then there are four distinct elements $a, b, c, d \in S$ such that $a + b = c + d$

Note: This problem is fairly difficult, in my opinion. I would say that it is representative of a difficult question on an exam, but not a typical question on an exam.

This is a pigeon-hole principle question. We can rephrase it as the following: Show that there are 2 disjoint subsets of S, $\{a,b\}$ and $\{c,d\}$ that have the same sum.

Let the pigeons be the size-2 subsets of S. There are $\binom{15}{2} = 105$ of these.

Let the holes be the possible sums. The minimum sum is $1+2=3$ and the maximum is $49+50=99$. So there are 97 possible sums.

Since we have more pigeons than holes, by pigeonhole principle there are 2 subsets of size 2 that have the same sum.

We now have to show that these 2 subsets are disjoint.

Suppose that $\{a,b\}$, $\{c,d\}$ are not disjoint. WLOG let $a=c$. Since $a+b=c+d$, this implies $b=d$. Which contradicts the fact that $\{a,b\}$, $\{c,d\}$ are two different subsets with the same sum.

5. (6 points) Find the number of onto functions from the set $\underbrace{\{1,2,3,4,5,6\}}^{S_1}$ to the set $\overbrace{\{a,b,c\}}^{S_2}$.

We can do this using inclusion-exclusion to compute the number of functions that are not onto.

Let:

A: be the set of functions from S_1 to S_2 s.t. a is not in the range.

B: be the set of functions from S_1 to S_2 s.t. b is not in the range.

C: be the set of functions from S_1 to S_2 s.t. c is not in the range.

The set of functions that are not onto is $A \cup B \cup C$.

By inclusion exclusion, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Note: $|A| = |B| = |C| = 2^6$ which is # of functions from a 6-element set to a 2 element set.

$|A \cap B| = |A \cap C| = |B \cap C| = 1^6 = 1$ which is # functions from a 6 element set to a 1-element set.

$|A \cap B \cap C| = 0$ since there are no functions from a set of 6-elements to the empty set.

(since every dot in the domain needs an arrow emanating from it).

So the set of functions that are not onto has cardinality $3 \cdot 2^6 - 3$.

The total # of functions from S_1 to $S_2 = 3^6$.

So the # of onto functions is $3^6 - 3 \cdot 2^6 + 3$

6. (5 points) Find integers y and z such that $55y + 38z = 1$.

We first do the Euclidean algorithm!

$$55 = \underline{38} \cdot 1 + \underline{17}$$

$$38 = \underline{17} \cdot 2 + \underline{4}$$

$$17 = 4 \cdot \underline{4} + \underline{1}$$

We now go backwards:

$$1 = 17 - 4 \cdot 4 = 17 - 4 \cdot (38 - 17 \cdot 2) = 9 \cdot 17 - 4 \cdot 38 =$$

$$-4 \cdot 38 + 9(55 - 38) = 9 \cdot 55 - 13 \cdot 38.$$

So $y = 9$, $z = -13$.

$$\begin{array}{r} \overset{4}{55} \\ \underline{9} \\ 495 \end{array}$$

$$\begin{array}{r} \overset{2}{38} \\ \underline{13} \\ 380 \\ \underline{114} \\ 494 \end{array}$$

7. (5 points) Factor the binomial coefficient $\binom{18}{7}$ as a product of primes.

$$\binom{18}{7} = \frac{18!}{7!\,11!} = \frac{\cancel{18}^{3} \cdot 17 \cdot 16 \cdot \cancel{15}^{\cancel{3}} \cdot \cancel{14}^{\cancel{2}} \cdot 13 \cdot \cancel{12}^{3}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot \cancel{7}} =$$

$2^4\ 3^2 \cdot 13 \cdot 17$