Math N55– Practice Midterm 2
Discrete Mathematics

Instructor: Max Hlavacek

July 30, 2019

Name: Sample

Student ID number: 1234567

In this exam, you are allowed writing utensils and one 8.5x11 notesheet. No calculators are allowed. Please write your name and id number at the top of each page in the space provided.

This exam contains 7 pages (including this cover page) and 5 questions. The total number of points is 50.
Good luck!

Distribution of Marks

Question	Points	Score
1	10	Good
2	10	Luck
3	10	!
4	10	!
5	10	!
Total:	50	!

1. Mark each of the following True or False. No explanation [illegible] ired. To ease the grading process, please clearly write T for true and F for false on the pro[illegible] d line.

(a) (2 points) F If X and Y are independent random [illegible] ables, then $E(XY) = E(X) + E(Y)$

(b) (2 points) F If X is a random variable on the sa[illegible] le space S, then $X(s) \geq 0$ for all $s \in S$.

(c) (2 points) T If E and F are independent events i[illegible] probability space, then E and $\bar{F}$ are also independent.

(d) (2 points) F Suppose we have holes labeled $0, 1, .$[illegible] n. If we put $n + 1$ objects in the holes, then at least one hole will contain more than [illegible] e object.

(e) (2 points) F The recurrence $f_n = f_{n-2}$ where $f(0)$ [illegible] 1 has exactly one solution.

2. (10 points) A store gives out g t certificates in the amounts of 10 and 25. What amounts of money can you make using gi certificates from the store? Prove your answer using strong induction. Note: You may hav to do a finite number of cases separately.

You can make the following amounts:

$\{10, 20, 25, 30, 35, 40, 45, 50, \dots\}$

In other words, it seems as though you can make $\$10$ and $\$5n$ for $n \geq 4$.

We can see that we can make $10 by using just one $10 gift certificate. We will prove the strong statement: We can make $\$5n$ for $n \geq 4$ by strong induction.

We have 2 base cases:

for, $n = 4$ you can make $20 using 2 $10 certificates,

$n = 5$ you can make $25 using a $10 certificate and a $15 certificate.

For our strong inductive hypothesis, we assume for some k, you can make $\$5j$ using gift certificates for every $j \leq k$.

Now, consider $\$5(k+1)$. Note that $5(k+1) = 5(k-1) + 10$.

By our strong inductive hypothesis, you can make $\$5(k-1)$ with gift certificates, and the extra $10 can be added with another $10 gift certificate. So you can make $\$5(k+1)$.

So by strong induction, you can make $\$5n$ for $n \geq 4$.

3. Answer the following questions about the set of bit strings of length n.

   (a) (2 points) For $n \geq 0$, determine the number of bit strings of length n that do not contain the string 01.

   (b) (4 points) Let b_n be the number of bit strings of length n that do not contain 000. Show that $b_0 = 1, b_1 = 2$, and $b_2 = 4$ and that

$$b_{n+3} = b_{n+2} + b_{n+1} + b_n$$

   (c) (4 points) Find the number of strings of length 10 with 4 ones and 6 zeros that do not contain the string 11.

(a). There are $n+1$ of these. Since 01 is not in the string, all the 1's (if they exist) need to be before the 0's (if they exist). So we can have

$$\underbrace{000\ldots0}_{n}, \quad \underbrace{1000\ldots0}_{n}, \quad \underbrace{1100\ldots0}_{n}, \quad \ldots, \quad \underbrace{1111\,1\ldots1}_{n}$$

You can verify there are $n+1$ of these.

(b). Consider a bit string of length $n+3$ that does not have 000. We do some casework.

Case 1: The string ends in 1. Then there are b_{n+2} ways to choose the rest of the digits.

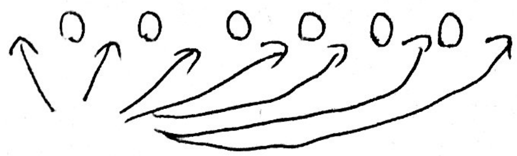
Case 2: The string ends in a 0: we further split into cases:

Case 2.1 The string ends in 10. There are b_{n+1} ways to choose rest of string.

Case 2.2. The string end in 00. Since there aren't 3 consecutive 0's it must end in 100. So there are b_n ways to fill in the rest.

So $b_{n+3} = b_{n+2} + b_{n+1} + b_n$ as desired.

(c.) Therefore we arrange the 6 zeros:



There are 7 slots in which to put the 4 ones (and each slot can only hold at most one 1).

So there are $\binom{7}{4}$ ways.

4. (a) (5 points) I have a bag of coins. Nine are fair coins, and the 10th has a heads on both sides. I draw a coin at random and flip it. If the coin comes up heads, what is the probability that the coin was a 2-headed coin?

(b) (5 points) Let X be a random variable, and $a \in$ R ← sorry haha. Show that $V(aX) = a^2V(X)$.

(a). Let E be the event that it comes up head, and F be event that the coin was 2-head.

We want to find $p(F|E)$. It is not immediately clear how to do this, so we use Baye's thm:

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$$

Note that $p(E|F) = 1$, $p(F) = \frac{1}{10}$,

$p(E|\bar{F}) = \frac{1}{2}$, $p(\bar{F}) = \frac{9}{10}$.

So $$p(F|E) = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{9}{20}} = \frac{\frac{1}{10}}{\frac{11}{20}} = \frac{2}{11}.$$

(b) Remember that

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$

So $$V(aX) = \sum_{s \in S} (aX(s) - E(aX))^2 p(s) =$$

$$\sum_{s \in S} (aX(s) - aE(X))^2 p(s) = \sum_{s \in S} a^2 (X(s) - E(X))^2 p(s)$$

$$= a^2 \sum_{s \in S} (X(s) - E(X))^2 p(s) = a^2 V(X)$$ as desired.

5. (10 points) Show using a combinatorial proof that:

$$\binom{2n}{3} = \binom{n}{3} + \binom{n}{3} + \binom{n}{2}n + n\binom{n}{2}$$

We do a combinatorial proof by counting the number of ways to choose 3 animals from a set of n goats and n sheep. (where each individual animal is distinguishable)

Way 1: We have a total of $2n$ animals, and need to choose 3. There are $\binom{2n}{3}$ ways to do this

Way 2: We have 4 cases.

Case 1: We choose 3 sheep There are $\binom{n}{3}$ ways to do this

Case 2: We choose 3 goats There are $\binom{n}{3}$ ways to do this.

Case 3: We choose 2 sheep and 1 goat. There are $\binom{n}{2}$ ways to choose the sheep and n ways to choose the goat. so $\binom{n}{2}n$ ways in total.

Case 4: We choose 1 sheep and 2 goats There are n ways to choose the sheep and $\binom{n}{2}$ ways to choose the goat

So adding the cases together, there are $\binom{n}{3} + \binom{n}{3} + \binom{n}{2}n + n\binom{n}{2}$ ways to choose the 3 animals

Since they count the same quantity,

$$\binom{2n}{3} = \binom{n}{3} + \binom{n}{3} + \binom{n}{2}n + n\binom{n}{2}$$