Homework Sections 5.2-5.4, 2.4

Max's lecture
MATH 55

Due July 11, 2019

Note: All problems are taken from Rosen, Discrete Mathematics and its applications, 8th ed. Have fun, and please feel free to ask each other and me for help!

Exercise 5.1.10. Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

by examining the values of this expression for small values of n. Then prove the formula you conjectured.

Exercise 5.1.20. Prove that $3^n < n!$ if n is an integer greater than 6.

Exercise 5.1.46. Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

Exercise 5.1.63. (NOTE: THIS PROBLEM IS EXTRA CREDIT!) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The arithmetic mean of these numbers is defined by:

$$A = (a_1 + \cdots + a_n)/n$$

and the geometric mean of these numbers is defined by

$$G = (a_1 a_2 \ldots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$

Exercise 5.1.51. What is wrong with the following proof:
Theorem: For every positive integer n, if x and y are positive integers with $\max(x, y) = n$ then $x = y$.
Basis step: Suppose $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, then $x = 1$ and $y = 1$.
Inductive step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the hypothesis,$x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

Exercise 5.2.3. Let $P(n)$ be the statement that a postage of n stamps can be formed using just 3 and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 8$.

(a) Show that the statements $P(8), P(9)$ and $P(10)$ are true, completeing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$.

(b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$.

(c) What do you need to prove in the inductive step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 8$?

(d) Complete the inductive step for $k \geq 10$. Explain why these steps show that $P(n)$ is true whenever $n \geq 8$.

Exercise 5.2.10. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. THe entire bar, or any smaller rectangular piece of the bar, can be broken along a vertical or horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

Exercise 5.2.14. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones respectively you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n-1)/2$.

Exercise 5.2.32. Find the flaw with the following "proof" that every postage of three cents or more can be formed using just $3-$ and $4-$ cent stamps.

Basis step: We can form postage of three cents with a single 3- cent stamp and we can form postage of 4 cents using a single 4-cent stamp.

Inductive step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just 3 and 4 cent stamps. We can then form postage of $k+1$ cents py replacing one 3-cent stamp with a 4-cent stamp or by replaying two 4-cent stamps by three 3-cent stamps.

Exercise 2.4.6bdf. List the first 10 terms of each of these sequences:

(b) the sequence whose nth term is the sum of the first n positive integers.

(d) The sequence whose nth term is $\lfloor \sqrt{n} \rfloor$.

(f) The sequence whose nth term is the largest integer whose binary expansion has n bits.

Exercise 2.4.12bd. Show that the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if:

(b) $a_n = 1$

(d) $a_n = 2(-4)^n + 3$

Exercise 2.4.30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$

(a) $\sum_{j \in S} j$

(b) $\sum_{j \in S} j^2$

(c) $\sum_{j \in S} (1/j)$

(d) $\sum_{j \in S} 1$

Exercise 2.4.34. Compute each of these double sums:

1. $\sum_{i=1}^{3} \sum_{j=1}^{2} (i - j)$
2. $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i + 2j)$
3. $\sum_{i=1}^{3} \sum_{j=0}^{2} (j)$
4. $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

Exercise 5.3.2d. Find $f(1), f(2), f(3), f(4)$, and $f(5)$ if f is defined recursvely by $f(0) = 3$ and for $n = 0, 1, 2, \ldots,$ we have that $f(n + 1) = 3^{f(n)/3}$

Exercise 5.3.8. Give a recursive defintion of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots,$ if

(a) $a_n = 4n - 2$

(b) $a_n = 1 + (-1)^n$

(c) $a_n = n(n + 1)$

(d) $a_n = n^2$

Exercise 5.3.12. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ where n is a positive integer.

Exercise 5.3.17. Determine the number of divisions used by the Euclidean algorithm to find the greatest common divisor of the fibonacci numbers f_n and f_{n+1}, where n is a non-negative integer. Verify your answer using mathematical induction.

Exercise 5.4.14. Give a recursive algorithm for finding the mode of a list of integers. (Note: This will not be graded for correctness! Don't spend too much time on this one, since I could not find any recursive algorithm that felt really satisfying.)