Math N55– Practice Midterm 2
Discrete Mathematics

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Name: ______________________________

Student ID number: ______________________

In this exam, you are allowed writing utensils and one 8.5x11 notesheet. No calculators are allowed. Please write your name and id number at the top of each page in the space provided.

This exam contains 7 pages (including this cover page) and 5 questions. The total number of points is 50.
Good luck!

Distribution of Marks

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. Mark each of the following True or False. No explanation required. To ease the grading process, please clearly write T for true and F for false on the provided line.

   (a) (2 points) F If X and Y are random variables on the same sample space, then $V(X + Y) = V(X) + V(Y)$

   (b) (2 points) F There are m^n functions from a set of size m to a set of size n.

   (c) (2 points) T If X and Y are random variables on the same sample space, then $E(X + Y) = E(X) + E(Y)$

   (d) (2 points) T The coefficient of x^n in $(1+x+x^2)^6$ is the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = n$ where $x_1, x_2, x_3, x_4, x_5, x_6$ are all nonnegative integers less than 3.

   (e) (2 points) F Suppose we have a linear, constant-coefficient recurrence that is nonhomogenous. Let a_n be a particular solution, and b_n be a solution to the homogenous form of the recurrence. Then every solution of our original recurrence can be written as $a_n + \alpha b_n$ for some real number α.

2. (10 points) Use induction to prove that a set with n elements has $n(n-1)/2$ subsets of cardinality 2 whenever n is an integer greater than or equal to 3. (Note: Do not use the formula for the number of combinations in your proof. The goal is to prove this formula for $r = 2$).

We do this by induction. Base case: $n=3$

We verify that there are 3 subsets of size 2 of a set of size 3. Indeed, $\frac{3(3-1)}{2} = 3$.

Now suppose that our statement holds for some positive integer $k \geq 3$.

We now consider a set of size $k+1$.

We consider one special element, a.

When we construct subsets of size 2, we have 2 cases.

Case 1: a is in the subset. There are k choices for the other element.

Case 2: a is not in the subset. We choose both elements from the remaining k elements. By our inductive hypothesis, there are $\frac{k(k-1)}{2}$ ways to do this.

So all in all, there are $\frac{k(k-1)}{2} + k = \frac{k(k-1)}{2} + \frac{2k}{2}$

$= \frac{k(k-1+2)}{2} = \frac{k(k+1)}{2}$ ways to choose a subset of size 2, as desired.

3. Answer the following questions about recurrences. (The two parts are not related)

   (a) (6 points) Let a_n be the number of strings of length n with the following properties:

      1. Each digit can be 1,2, or 3
      2. The string has two consecutive 1's.

   Find a recurrence for a_n.

We have 2 cases:

Case1: string ~~[illegible]~~ does not start with a 1.

Then: there are a_{n-1} ways to choose the rest of the digits so there are 2 cons. 1's.

There are 2 ways to choose the first digit. So this case contribute $2a_{n-1}$.

Case 2: The string does start with a 1. Then we split into 2 subcases.

Case 2.1 2nd digit is not a 1. similar to case 1, this case contributes $2a_{n-2}$.

Case 2.2 2nd digit is a 1. Then the remaining $n-2$ digits have no restrictions, so this case contributes 3^{n-2}.

Overall: $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$.

   (b) (4 points) Suppose we have a linear, constant-coefficient recurrence whose characteristic equation has roots $-1, 1, 1, 4, 4, 5, 5, 5$. Write down the set of all solutions of this recurrence.

I left the word homogenous out, so I did not include this question in final calculation of the grade.

If I did include the word homogenous, the answer is

$$\alpha_1(-1)^n + \alpha_2(1)^n + \alpha_3 n(1)^n + \alpha_4 (4)^n + \alpha_5 n(4)^n + \alpha_6(5)^n + \alpha_7 n(5)^n + \alpha_8 n^2 (5)^n.$$

4. (a) (5 points) Let X be a random variable, and $a \in \mathbb{R}$. Show that $V(X+a) = V(X)$

There are several ways to show this. One way is:

$V(X+a) =$

(b) (5 points) Suppose we perform n trials, where each trial consists of rolling 2 die. Let X be defined as 2 plus the number of trials in which the two die sum to 6. Use part (a) to compute $V(X)$.

This is a Binomial dist. problem, where our success is that the sum of the dice is 6.

The prob. that this happens is $\frac{5}{36}$. So our parameters for our binomial dist are n, $p = \frac{5}{36}$, $q = 1-p = \frac{31}{36}$.

We know that the variance of a binom dist is npq.

So $V(X) = V(Y-2) = V(Y) = npq = n \cdot \frac{5 \cdot 31}{36^2} = \frac{155}{36^2} n$.

where Y is the bin. distribution given by counting the number of times the dice sum to 6 without adding 2.

5. (a) (5 points) Prove that if you choose 5 distinct numbers from the set $\{1,2,3,4,5,6,7,8\}$, you are guaranteed to have a pair whose sum is 9.

This is a pigeon-hole principle problem. The "pigeons" are the 5 distinct numbers we choose, and the "holes" are

$\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$ Two numbers must be in the same hole, and thus must sum to 9.

(b) (5 points) How many subsets of $\{1,2,3,4,5,6,7,8\}$ contain 1,2, or 3? No need to simplify your answer.

There are several ways to do this. One way, using inclus./exclus., is to let.

$A = \{\text{subsets containing } 1\}$

$B = \{\text{subsets containing } 2\}$

$C = \{\text{subsets containing } 3\}$.

So $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| =$

$3 \cdot 2^7 - 3 \cdot 2^6 + 2^5 = 224.$

This page is intentionally left blank to accommodate work that wouldn't fit elsewhere and/or scratch work.