k-Schur functions, graded S_n modules, and the flag variety

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Catabolism

Definition

A semistandard tableau T of partition weight μ is *t-catabolizable* if entries 1 through t occupy shape $(\mu_1, \ldots, \mu_t)$.

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Example: A 3-catabolizable tableau.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
5 \\
\cline{1-1}
4 & 4 & 6 & 6 \\
\cline{1-4}
3 & 3 & 3 & 5 & 7 \\
\cline{1-5}
2 & 2 & 2 & 2 & 4 \\
\hline
1 & 1 & 1 & 1 & 1 & 5 & 6 & 7 & 8 \\
\hline
\end{array}$$

Definition

If catabolizable, we *t-catabolize* T as follows.

First split T at the t-th row...

5			
4	4	6	6

3	3	3		
2	2	2	2	
1	1	1	1	1

5	7				
	4				
		5	6	7	8

Definition

If catabolizable, we *t-catabolize* T as follows.

Drop entries 1 through $t\ldots$

$$\begin{array}{cccc} 5 & & & \\ 4 & 4 & 6 & 6 \end{array}$$

$$\begin{array}{cccccc} 5 & 7 & & & & \\ & 4 & & & & \\ & & 5 & 6 & 7 & 8 \end{array}$$

Definition

If catabolizable, we *t-catabolize* T as follows.

Swap the upper and lower parts. . .

$$\begin{array}{cccc} 5 & & & \\ 4 & 4 & 6 & 6 \end{array} \qquad \begin{array}{cccccc} 5 & 7 & & & & \\ & 4 & & & & \\ & & 5 & 6 & 7 & 8 \end{array}$$

Definition

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Swap the upper and lower parts. . .

$$
\begin{array}{|c|cccc}
\cline{1-1}
5 & \multicolumn{4}{c}{} \\
\hline
4 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{6} \\
\hline
\end{array}
\qquad
\begin{array}{cccccc}
\cline{1-2}
\multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} & & & & \\
\cline{1-2}
& \multicolumn{1}{|c|}{4} & & & & \\
\cline{2-6}
& & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\
\cline{3-6}
\end{array}
$$

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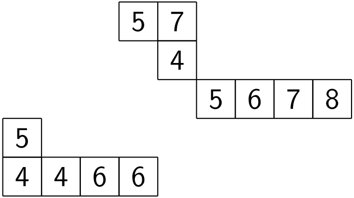

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Swap the upper and lower parts...

$$\begin{array}{|c|c|cccc}
\hline
5 & 7 & \multicolumn{4}{c}{} \\
\hline
\multicolumn{1}{c|}{} & 4 & \multicolumn{4}{|c}{} \\
\cline{2-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\
\cline{3-6}
\end{array}$$

$$\begin{array}{|c|c|c|c|}
\cline{1-1}
5 & \multicolumn{3}{l}{} \\
\hline
4 & 4 & 6 & 6 \\
\hline
\end{array}$$

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5	7			
	4			
	5	6	7	8

5			
4	4	6	6

Definition

If catabolizable, we *t-catabolize* T as follows.

Then rectify by *jeu de taquin*.

5	7							
	4							
	5	6	7	8				
					5			
					4	4	6	6

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6				
5	5	5	7	8
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$$\begin{array}{ccccc} 7 & & & & \\ 6 & & & & \\ 5 & 5 & 5 & 7 & 8 \\ 4 & 4 & 4 & 6 & 6 \end{array}$$

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6	6	7
		6

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	6	6

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6				
5	5	5	7	8
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is itself 2-catabolizable.
If we then 2-catabolize it

8		
7	7	
6	6	6

we get a 3-catabolizable tableau.

Therefore, we say that the tableau we started with

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\cline{1-1}
5 \\
\cline{1-4}
4 & 4 & 6 & 6 \\
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3 & 3 & 3 & 5 & 7 \\
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is $(3, 2, 3)$-catabolizable.

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More generally,

Definition

Given a composition $\tau = (\tau_1, \tau_2, \ldots, \tau_k)$, a tableau T of partition weight is *τ-catabolizable* if it is τ_1-catabolizable, and, inductively, its τ_1-catabolism is $(\tau_2, \ldots, \tau_k)$-catabolizable.

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A related operation is *cyclage*: row-uninsert any entry $x > 1$ from T, then column-insert x into the tableau S that remains.

Example:

$$T = \begin{array}{cccc} 3 & 4 & & \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{array} \qquad S = \begin{array}{cccc} 3 & & & \\ 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 3 \end{array} \qquad T' = \begin{array}{ccccc} 3 & & & & \\ 2 & 2 & 2 & 4 & \\ 1 & 1 & 1 & 3 & 4 \end{array}$$

$$x = \boxed{2}$$

Cyclage decreases charge by 1, *i.e.*, $c(T') = c(T) - 1$. (The assertion about catabolism follows from this and the *jeu de taquin* invariance of charge.)

Many elementary properties of catabolism are still unproven.

Conjecture

- *If σ is a refinement of τ, and T is τ-catabolizable, then T is σ-catabolizable.*

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- *If σ is a refinement of τ, and T is τ-catabolizable, then T is σ-catabolizable.*
- *If T' is a cyclage of T, and T' is τ-catabolizable, then so is T.*
- *(L-C. Chen) The 1-catabolism of T and its "1-rolling" admit the same catabolizabilities.*
- *The number of τ-catabolizable tableaux of weight μ and shape λ is the Littlewood-Richardson coefficient*

$$c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(k)}},$$

where $\mu^{(1)}$ is the first τ_1 parts of μ, $\mu^{(2)}$ is the next τ_2 parts, and so on.

Vector bundles on the flag variety

Some Notation

Group		Lie algebra
G	general linear group GL_m	$\mathfrak{gl}_m$
B	upper triangular matrices	$\mathfrak{b}$
U	unipotents in B	$\mathfrak{n}$

$G/B =$ flag variety. $T^*(G/B) = G \times_B \mathfrak{n} =$ cotangent bundle.

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$G/B =$ flag variety. $T^*(G/B) = G \times_B \mathfrak{n} =$ cotangent bundle. We have

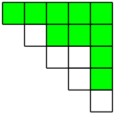
$$G\text{-invariant sub-bundles } G \times_B \mathfrak{j} \subseteq T^*(G/B)$$

$\leftrightarrow$ B-submodules $\mathfrak{j} \subseteq \mathfrak{n}$

$\leftrightarrow$ upper order ideals of positive roots

$\leftrightarrow$ partitions $\subseteq (m-1, m-2, \ldots, 1)$

Example (in $\mathfrak{gl}_6$):

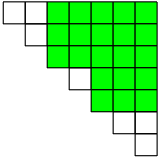
$\mathfrak{j} =$ 

To each composition τ of m corresponds a parabolic subgroup $P_\tau \subseteq GL_m$ of block upper triangular matrices.

The pullback of $T^*(G/P_\tau)$ is $G \times_B \mathfrak{j}$ for $\mathfrak{j} = \mathfrak{n}(P_\tau) =$ block strictly upper triangular matrices.

Example (in $\mathfrak{gl}_8$):

$$\mathfrak{n}(P_{(3,2,3)}) =$$



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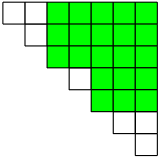
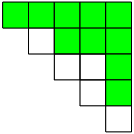


Definition

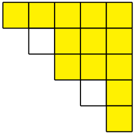
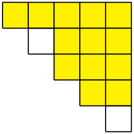
A tableau T is $\mathfrak{j}$-*catabolizable* if T is τ-catabolizable for every τ such that $\mathfrak{j} \subseteq \mathfrak{n}(P_\tau)$

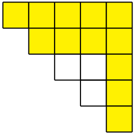
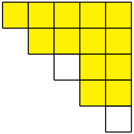
Assuming the refinement conjecture holds, τ-catabolizability is equivalent to $\mathfrak{n}(P_\tau)$-catabolizability.

Example: For

$$\mathfrak{j} =$$  ,

T is $\mathfrak{j}$-catabolizable iff it admits the τ-catabolizabilities below (and their refinements):

$\tau = (1, 2, 2, 1)$  , $\tau = (1, 2, 1, 2)$  ,

$\tau = (1, 1, 3, 1)$  , $\tau = (1, 1, 2, 2)$  .

Conjectured generalization of the Borel–Weil–Bott theorem

Let $\mathbb{C}_\mu$ = 1-dimensional B-module with weight μ.

Let w_0 denote the longest element in the Weyl group, *i.e.*, the permutation $12\ldots m \;\mapsto\; m\ldots 21$.

Recall the classical theorem of Borel, Weil, Bott.

Theorem

For μ a dominant weight of G, the line bundle $\mathcal{L}_\mu = G \times_B \mathbb{C}_{w_0(\mu)}$ satisfies

1. $H^0(G/B, \mathcal{L}_\mu)$ *is the irreducible G-module V_μ with highest weight μ.*
2. $H^i(G/B, \mathcal{L}_\mu) = 0$ *for $i > 0$.*

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For $G = GL_m$, dominant weights are essentially partitions with at most m parts, and the character of V_μ is the Schur function $s_\mu(x_1, \dots, x_m)$.

Theorem (from previous slide)

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Conjecture

For μ a dominant weight of GL_m, the pullback of $\mathcal{L}_\mu$ to $G \times_B \mathfrak{j}$ satisfies

1. $H^0(G \times_B \mathfrak{j}, \mathcal{L}_\mu)$ *has graded character*

$$\sum_{\lambda} \sum_{\substack{T \in SSYT(\lambda,\mu) \\ T \text{ is } \mathfrak{j}\text{-catabolizable}}} q^{c(T)} s_\lambda(\mathbf{x})$$

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In the case $\mathfrak{j} = \mathfrak{n}(P_\tau)$, Conjecture (1) is due to Shimozono–Weyman, and Conjecture (2) to Broer.

Proposition

Assuming the preceding conjecture holds, the generating function for $\mathfrak{j}$-catabolizable tableaux of weight μ is given by the formula

$$\sum_{\lambda} \sum_{\substack{T \in SSYT(\lambda,\mu) \\ T \text{ is } \mathfrak{j}\text{-catabolizable}}} q^{c(T)} s_\lambda(\mathbf{x}) = \Psi \left(\prod_{e_{ij} \in \mathfrak{j}} \frac{1}{1 - q\, x_i/x_j}\, x^\mu \right)_{\mathrm{pol}},$$

where

$$\Psi\, x^\nu = \begin{cases} (-1)^{l(w)} s_{w(\nu+\rho)-\rho} & \text{for } w(\nu+\rho) \text{ regular and dominant} \\ 0 & \text{if } \nu+\rho \text{ is not regular.} \end{cases}$$

Here $\rho = (m-1, m-2, \ldots, 0)$.

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Here $\rho = (m-1, m-2, \ldots, 0)$.

In any case, the right-hand side gives the graded character of the virtual representation $\sum_i (-1)^i \left[H^i(G \times_B \mathfrak{j}, \mathcal{L}_\mu) \right]$.

Li-Chung Chen's modules

Definition

Partitions μ and λ are *skew linked*, written $\mu \xrightarrow{\theta} \lambda$, if there exists a skew diagram θ with the same row lengths (in order) as μ and the same column lengths as λ.

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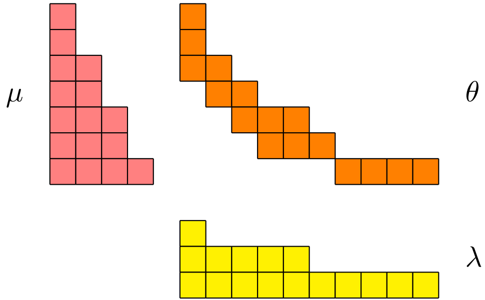
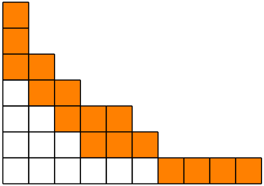
Example:



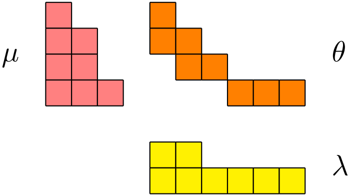
Proposition (L-C. Chen)

Suppose $\mu \xrightarrow{\theta} \lambda$ are partitions of n. Let $d = |\beta|$, where $\theta = \alpha/\beta$ (example below).

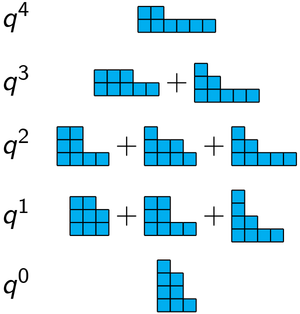
There is a unique graded, S_n-equivariant $\mathbb{C}[y_1, \ldots, y_n]$-module $\mathcal{M}_{\mu,\lambda}$, generated by its degree 0 part, which is irreducible with character χ^μ as an S_n-module, and co-generated by its degree d part, which is irreducible with character χ^λ.

$\theta =$  $d = 12$

Example:

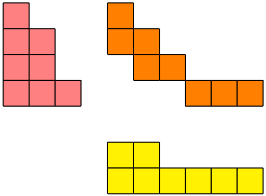


$\mathrm{ch}(\mathcal{M}_{\mu,\lambda})$



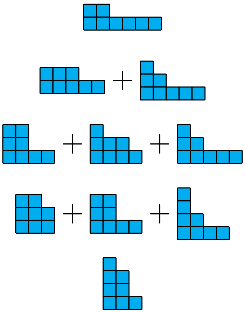
Example:

μ θ λ



$\mathrm{ch}(\mathcal{M}_{\mu,\lambda})$

q^4, q^3, q^2, q^1, q^0



In this example, θ is the outer zone of a *(3+1)-core*, λ' is the *3-conjugate* of the 3-bounded partition μ, and $\mathrm{ch}(\mathcal{M}_{\mu,\lambda})$ coincides with the Lascoux–Lapointe–Morse *3-atom* $A_{\mu}^{(3)}(\mathbf{x};q) = A_{(3,2,2,1)}^{(3)}(\mathbf{x};q)$.

Conjecture

Let μ be k-bounded and let λ' be its k-conjugate. (Recall that, by definition, this means $\mu \stackrel{\alpha/\beta}{\rightarrow} \lambda$, where α is a $(k+1)$-core and $\beta \subseteq \alpha$ is the set of boxes with hook-length $> k+1$.) Then we have

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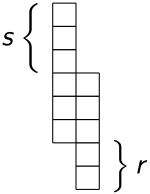
In fact, we conjecture that every Chen module $\mathcal{M}_{\mu,\lambda}$ has graded character matching that of $H^0(G \times_B \mathfrak{j}, \mathcal{L}_\mu)$ for a special choice of $\mathfrak{j}$.

If so, then as we have seen, we can conjecturally express this character both as a generating function for catabolizable tableaux, and by an explicit raising operator formula.

Chen's modules and the flag variety

Consecutive columns in a skew-linking shape θ always have $r \leq s$, with r and s as shown at right. Hence we can match the beginning of each row to the end of some higher row.



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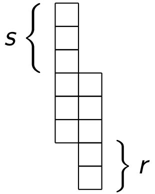

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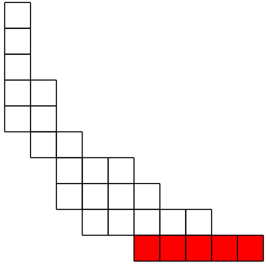

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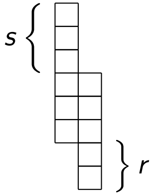

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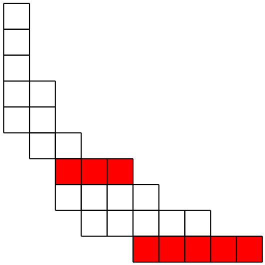

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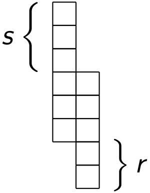

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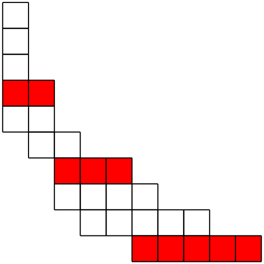

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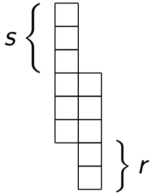

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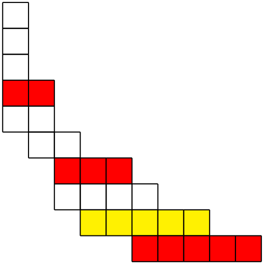

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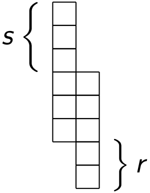

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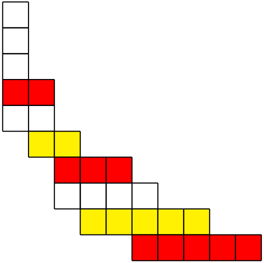

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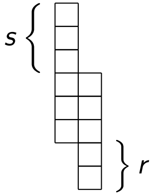

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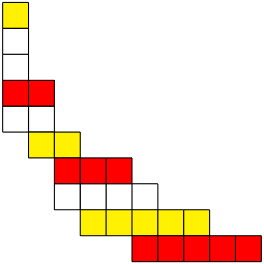

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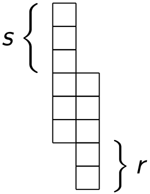

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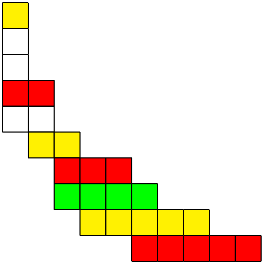

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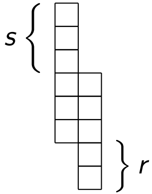

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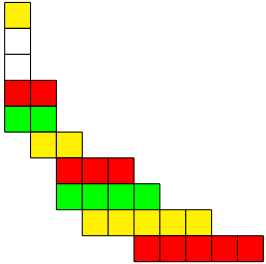

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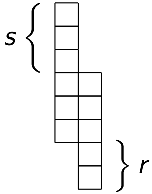

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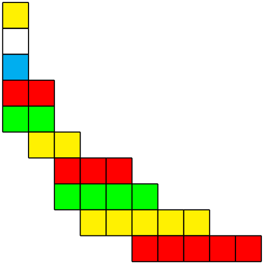

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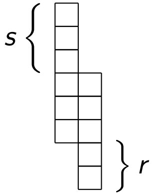

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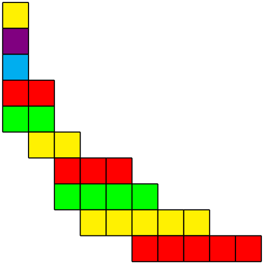

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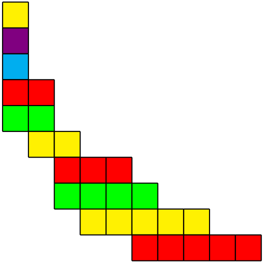

i	$f(i)$
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8	∞
7	∞
6	∞
5	10
4	7
3	6
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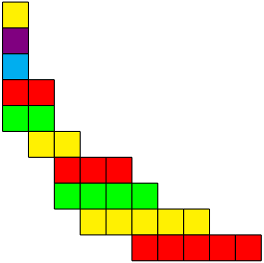

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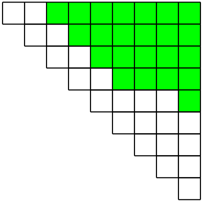
Then define $\mathfrak{j}(\theta) = \langle e_{ij} \mid j \geq f(i) \rangle$ (for this example, in $\mathfrak{gl}_{10}$).



Conjecture (L-C. Chen)

Suppose $\mu \xrightarrow{\theta} \lambda$. Under the Frobenius map $\chi^\lambda \mapsto s_\lambda(\mathbf{x})$, the graded S_n character of $\mathcal{M}_{\mu,\lambda}$ is given by the formula

$$\sum_{\lambda} \sum_{\substack{T \in SSYT(\lambda,\mu) \\ T \text{ is } \mathfrak{j}(\theta)\text{-catabolizable}}} q^{c(T)} s_\lambda(\mathbf{x}),$$

which we also conjecture is the graded GL_m character of $H^0(G \times_B \mathfrak{j}(\theta), \mathcal{L}_\mu)$.

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Remark: In the k-atom case, the constraint that T be $\mathfrak{j}(\theta)$-catabolizable is *a priori* stronger than the catabolizability constraint in the original definition by Lascoux, Lapointe and Morse. But it appears to pick out the same set of tableaux.

Define $\mathcal{N}_{\mu,\mathfrak{j}} = H^0(G \times_B \mathfrak{j}, \mathcal{L}_\mu)$. We expect the identity of characters conjectured on the previous slide to reflect a simple relationship between $\mathcal{M}_{\mu,\lambda}$ and $\mathcal{N}_{\mu,\mathfrak{j}(\theta)}$.

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Take $m = n$. Since $\mathcal{N}_{\mu,\mathfrak{j}(\theta)}$ is a GL_n-equivariant $\mathbb{C}[\mathfrak{gl}_n]$-module, its 0-weight space* is an S_n-equivariant $\mathbb{C}[y_1, \ldots, y_n]$-module.

[*for experts: we mean the 0-weight space with respect to SL_n of the polynomial part of this GL_n-module.]

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Conjecture

With the preceding definitions, the 0*-weight space of* $\mathcal{N}_{\mu,\mathfrak{j}(\theta)}$ *is isomorphic to* $\mathcal{M}_{\mu,\lambda}$.

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Conundrums

For fun, we'll finish by listing some of the "known" ways to compute $A_\mu^{(k)}(\mathbf{x}; q)$. To the best of my knowldege, no two of these have yet been proven equivalent.

(More is known when $q = 1$, but I won't discuss that here.)

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