A Representation-Theoretic Model for k-Atoms

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Skew-linked partitions

Definition

Partitions λ and μ are *skew linked*, written

$$\lambda \xrightarrow{\theta} \mu$$

if there exists a skew diagram θ with the same row lengths (in order) as λ and the same column lengths as μ.

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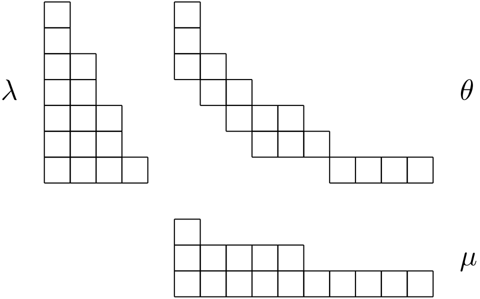

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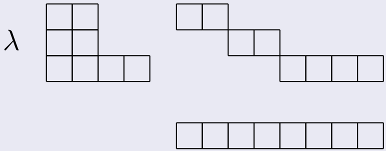

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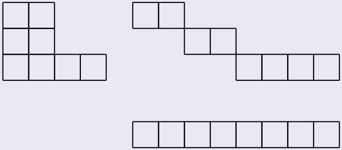

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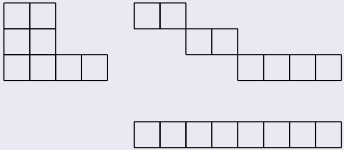

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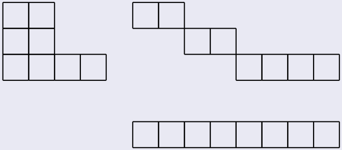

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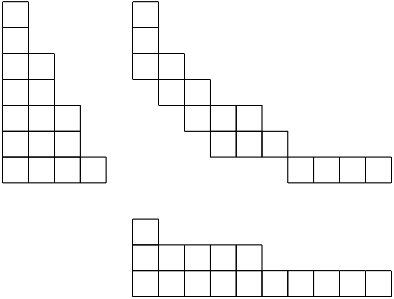
- Transpose symmetry: $\lambda \xrightarrow{\theta} \mu$ if and only if $\mu' \xrightarrow{\theta'} \lambda'$
- The two partitions λ and μ determine θ (and conversely, of course).

The "k-atom" case

Let κ be a $(k+1)$-*core* (no hook-length $= k+1$), and let θ be the set of boxes in κ with hook-length at most k.

Example ($k = 4$):

λ



θ

μ

Then θ skew-links a k-bounded partition λ to the transpose of its Lapointe-Morse *k-conjugate*:

$$\lambda \xrightarrow{\theta} \mu = (\lambda^{[k]})'$$

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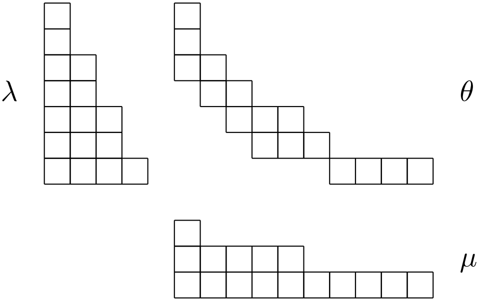
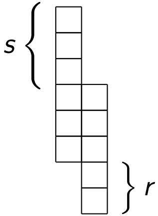


These cases, along with the $\mu = (n)$ cases, are the key examples!

Decomposing a skew-linking shape θ into row chains

Consecutive columns in a skew-linking shape θ always have $r \leq s$, with r and s as shown at right. Hence we can match the beginning of each row to the end of some higher row.



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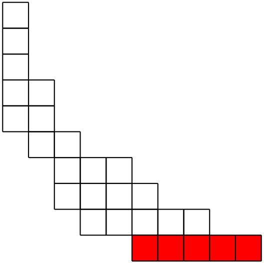

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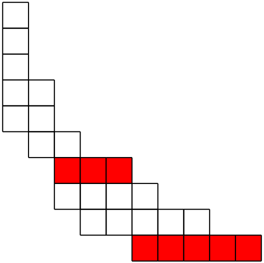

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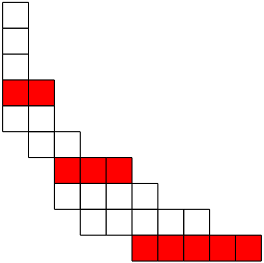

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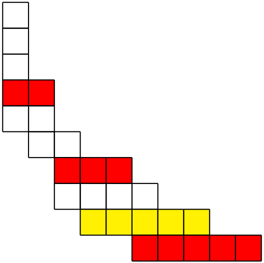

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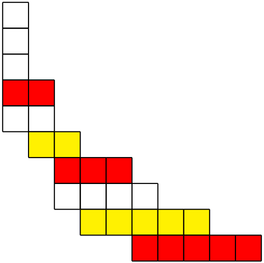

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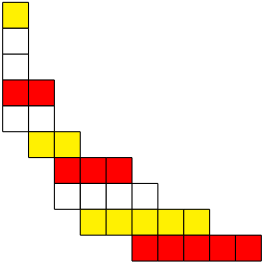

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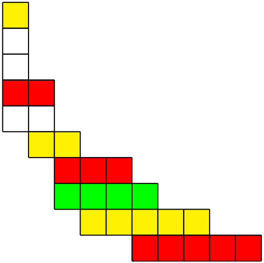

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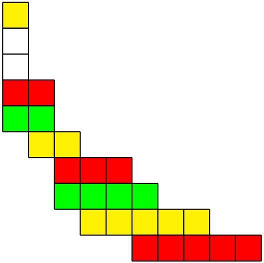

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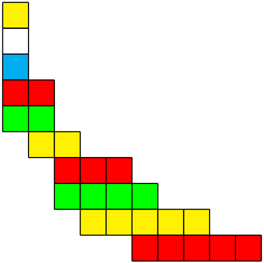

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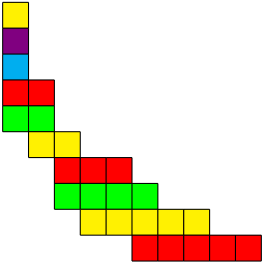

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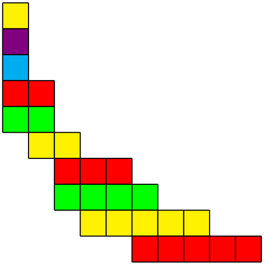
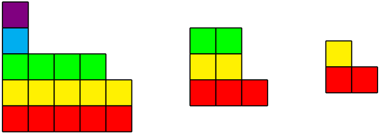


Now we group the rows into partitions, according to how far each row is from the end of its chain.



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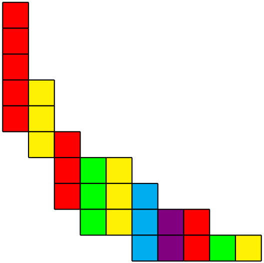
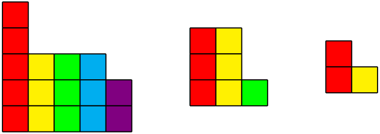


A remarkable fact is that doing it by columns leads to the same tuple of partitions.



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Some other (easy) facts

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  $$\nu^{(1)} \supseteq \nu^{(2)} \supseteq \cdots \supseteq \nu^{(r)}.$$

  In particular,

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- The statistic

  $$n(\gamma) \underset{\text{def}}{=} \sum_i (i-1)\,\gamma_i = \sum_i (i-1)\,|\nu^{(i)}|$$

  is equal to the number of "missing boxes," $|\beta|$, where $\theta = \alpha/\beta$.

How to construct small $\mathbb{C}[\mathbf{x}] * S_n$ modules

Note: "$\mathbb{C}[\mathbf{x}] * S_n$ module" $=$ "$\mathbb{C}[x_1, \ldots, x_n]$ module with S_n action."

Motivation: How to construct irreducible S_n-modules.

Let $V = \varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ be the S_n module induced from the sign representation of the Young subgroup $S_{\lambda'}$.

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The irreducible V_λ is the image of the essentially unique homomorphism

$$V \underset{\phi}{\longrightarrow} W.$$

This uniquely characterizes V_λ as

1. generated by an (essentially unique) $S_{\lambda'}$-antisymmetric element, and
2. *co-generated* by an (essentially unique) S_λ-invariant linear functional.

Question

Which $\mathbb{C}[\mathbf{x}] * S_n$ modules can be characterized in a similar fashion?

Let $V = \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]$, the free $\mathbb{C}[\mathbf{x}]$ module on our previously considered induced S_n module.

Let $W = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$, a co-free $\mathbb{C}[\mathbf{x}]$ module on an induced S_n module, but we may have $\mu \neq \lambda$.

Let d be the smallest degree such that there is a non-zero S_n-module homomorphism

$$\psi\colon \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_d \to 1 \uparrow_{S_\mu}^{S_n} .$$

Suppose further that λ and μ are such that ψ is essentially unique.

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Proposition

With the above hypotheses, there is an essentially unique $\mathbb{C}[\mathbf{x}] * S_n$ homomorphism, homogeneous of degree zero

$$\phi\colon V \to W[-d].$$

Its image $M_{\lambda,\mu}$ is a graded $\mathbb{C}[\mathbf{x}] * S_n$ module uniquely characterized as

1. *generated by an (essentially unique) $S_{\lambda'}$-antisymmetric element (in degree 0), and*
2. *co-generated by an (essentially unique) S_μ-invariant linear functional (on the top degree, which is equal to d).*

Main theorem

Theorem (Chen)

1. *The necessary and sufficient condition for the hypotheses of the preceding proposition to hold is that λ be skew-linked to μ.*
2. *In that case, the degree $d =$ (top degree of $M_{\lambda,\mu}$) is equal to $n(\gamma) = |\beta|$, where the skew diagram linking λ to μ is $\theta = \alpha/\beta$.*
3. *Moreover, the degree zero and top degree components of $M_{\lambda,\mu}$ are irreducible S_n modules isomorphic to V_λ and V_μ, respectively.*

Special cases

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Remark: Garsia and Procesi prove the character formula directly from the structure of the module. Conceivably, we might determine the character of a general $M_{\lambda,\mu}$ by similarly elementary means.

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Conjecture

If λ is k-bounded and $\mu = (\lambda^{[k]})'$ is the transpose of its k-conjugate, then the graded Frobenius characteristic of $M_{\lambda,\mu}$ is equal to the k-atom $A_\lambda^{(k)}(z;t)$ of Lascoux, Lapointe and Morse.

Bits of the proof of Chen's theorem

Goal: characterize λ, μ such that the space

$$\mathrm{Hom}_{S_n}\left(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_d,\ 1 \uparrow_{S_\mu}^{S_n} \right)$$

has dimension 1 in the smallest degree d for which it is non-zero.

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Lemma

The desired degree d is the minimum of

$$\sum_{i,j} \binom{a_{i,j}}{2}$$

over all non-negative integer matrices A with row sums μ and column sums λ'.

The desired dimension-one condition holds if and only if the minimizing matrix A is unique.

Proposition (Chen)

A matrix A with specified, weakly decreasing, row and column sums uniquely minimizes $\sum_{i,j}\binom{a_{i,j}}{2}$ *iff it satisfies the following conditions:*

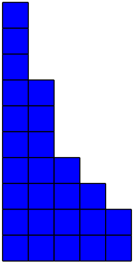
1. *The entries* $a_{i,j}$ *weakly decrease along rows and columns, i.e., A is a* plane partition*;*
2. *For every* 2×2 *minor* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *of A, we have*

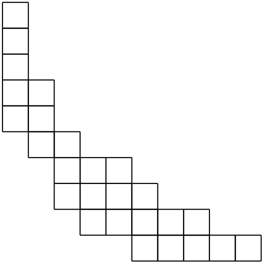
$$\begin{aligned} (a+d)-(b+c) &\le 1 \quad \text{if } a,d>0 \\ (b+c)-(a+d) &\le 1 \quad \text{if } b,c>0 \end{aligned}$$

Moreover, there exists such a matrix A with column sums λ' *and row sums* μ *if and only* λ *is skew-linked to* μ*, in which case A is given by the plane partition with layers* $(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(r)})$.

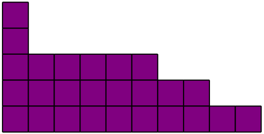
Example:

λ

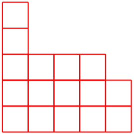




$$A = \begin{array}{cc} \begin{pmatrix} 3 & 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} 10 \\ 8 \\ 6 \\ 1 \\ 1 \end{array} \\ \begin{array}{ccccc} 10 & 7 & 4 & 3 & 2 \end{array} & \end{array}$$



μ



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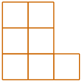

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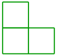

$\nu^{(3)}$

For the extra conclusion of Chen's theorem, that $M_{\lambda,\mu}$ is generated by V_λ and co-generated by V_μ, we must also prove that

$$\langle \chi^\lambda \otimes \mathrm{ch}(\mathbb{C}[\mathbf{x}]_d),\, \chi^\mu \rangle \neq 0.$$

This follows from

1. $d = \sum_i (i-1)\,|\nu^{(i)}|$, and
2. the Littlewood-Richardson coefficients $c^\lambda_{\nu^{(1)},\ldots,\nu^{(r)}}$ and $c^\mu_{\nu^{(1)},\ldots,\nu^{(r)}}$ are both non-zero.

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In fact, this implies that

$$\langle \chi^\lambda \otimes \chi^\gamma,\, \chi^\mu \rangle \;=\; 1.$$

Note that λ', μ and γ are the three projections of the plane partition given by the matrix A.

More conjectures about the modules in the k-atom case

Notation: We write $M_\lambda^{(k)}$ for $M_{\lambda,\mu}$ when λ is a k-bounded partition and $\mu = (\lambda^{[k]})'$ is the transpose of its k-conjugate.

Conjecture

The module $M_\lambda^{(k)}$ has an S_n equivariant resolution

$$F_m \to \cdots \to F_1 \to F_0 \to M_\lambda^{(k)} \to 0$$

by free $\mathbb{C}[\mathbf{x}]$ modules F_i whose generators contain only those irreducible S_n modules V_κ for which κ is k-bounded.

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Notation: $\mathcal{M}^{(k)}$ denotes the category of $\mathbb{C}[\mathbf{x}] * S_n$ modules having a k-bounded free resolution in the above sense.

Recall that the k-atoms $A_\lambda^{(k)}(z;t)$ should be a basis of the space $\Lambda^{(k)}$ spanned by the transformed Schur functions $S_\kappa[X/(1-t)]$, where κ is k-bounded.

Lemma

The graded Frobenius characteristic of any module M in $\mathcal{M}^{(k)}$ belongs to the space $\Lambda^{(k)}$.

Irreducible modules in the category $\mathcal{M}^{(k)}$

Definition

A module $M \neq 0$ is *irreducible in* $\mathcal{M}^{(k)}$ if M has no proper non-zero submodule in $\mathcal{M}^{(k)}$.

$\mathcal{M}^{(k)}$ is not an abelian category.

However, if two modules in a short exact sequence

$$0 \to K \to M \to Q \to 0$$

belong to $\mathcal{M}^{(k)}$, then so does the third. Hence:

Proposition

Every finite-length module M in $\mathcal{M}^{(k)}$ has a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

whose quotients M_i/M_{i-1} are irreducible in $\mathcal{M}^{(k)}$.

Conjecture

The irreducible modules in $\mathcal{M}^{(k)}$ are exactly the modules $M_\lambda^{(k)}$.

Consequences for properties of k-atoms

Our two conjectures—

(Let's recall them)

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1. *The modules* $M_\lambda^{(k)}$ *are the irreducibles in* $\mathcal{M}^{(k)}$*;*
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- The k-atoms do indeed belong to the space $\Lambda^{(k)}$.

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—imply many expected or new properties of k-atoms. Here are some of the elementary consequences.

- The k-atoms do indeed belong to the space $\Lambda^{(k)}$.
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Consequences for properties of k-atoms

Our two conjectures—

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- The k-atom $A_{\lambda[k]}^{(k)}(z,t)$ is the conjugate of $t^d A_\lambda^{(k)}(z;1/t)$.

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Finally, we have a conjectured combinatorial formula for the graded Frobenius characteristic of any $M_{\lambda,\mu}$ in terms of *charge* and *catabolism*. It generalizes both the classical charge formula of Lascoux and Schützenberger for Hall-Littlewood polynomials, and the combinatorial definition of k-atoms by Lascoux, Lapointe and Morse.

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—But that's a topic for another day...