Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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Homework 8 Solutions

4.2 Ex. 10. No, (d) fails for $\mathbf{u} = [\,0\ 1\ {-1}\ 0\,]^T$.

4.2 Ex. 41, 42. Take $\mathbf{u} = [\,1\ 1\ \dots\ 1\,]$, $\mathbf{v} = [\,a_1\ a_2\ \dots\ a_n\,]^T$. Then the left-hand side of the given inequality is $(\mathbf{u}\cdot\mathbf{v})^2$, while the right-hand side is $||\mathbf{u}||^2||\mathbf{v}||^2$.

4.3, Ex. 4. Errors are 32, 32, 16, and approximately 19.9. The best of the given lines is $m = 1$, $b = 0$. (The true least-squares best line, however, is not one of the given lines, but rather is $m = 11/15$, $b = 0$.)

4.3, Ex. 20. The projection matrix is

$$P = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} & \frac{1}{5} & -\left(\frac{1}{5}\right) \\ \frac{2}{5} & \frac{11}{35} & -\left(\frac{2}{35}\right) & -\left(\frac{8}{35}\right) \\ \frac{1}{5} & -\left(\frac{2}{35}\right) & \frac{29}{35} & \frac{11}{35} \\ -\left(\frac{1}{5}\right) & -\left(\frac{8}{35}\right) & \frac{11}{35} & \frac{9}{35} \end{bmatrix}.$$

The projected vector is

$$\mathbf{p} = \begin{bmatrix} \frac{8}{5} \\ \frac{29}{35} \\ \frac{52}{35} \\ -\left(\frac{2}{35}\right) \end{bmatrix}.$$

Problem A. (a) Recall the formula

$$F_k = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k), \quad \text{where} \quad \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}.$$

Then

$$\begin{aligned} 2F_{k+1}F_k - F_k^2 &= \frac{2(\alpha^{k+1} - \beta^{k+1})(\alpha^k - \beta^k) - (\alpha^k - \beta^k)^2}{5} \\ &= \frac{(2\alpha - 1)\alpha^{2k} + (2\beta - 1)\beta^{2k} - (2\alpha + 2\beta - 2)(\alpha^k\beta^k)}{5}. \end{aligned}$$

Using the facts that $2\alpha - 1 = \sqrt{5}$, $2\beta - 1 = -\sqrt{5}$, and $\alpha + \beta = 1$, this simplifies to F_{2k}.

Next,

$$\begin{aligned} F_{k+1}^2 + F_k^2 &= \frac{(\alpha^{k+1} - \beta^{k+1})^2 + (\alpha^k - \beta^k)^2}{5} \\ &= \frac{(\alpha + \alpha^{-1})\alpha^{2k+1} + (\beta + \beta^{-1})\beta^{2k+1} - (2\alpha\beta + 2)\alpha^k\beta^k}{5}. \end{aligned}$$

Now using the facts that $\alpha + \alpha^{-1} = \sqrt{5}$, $\beta + \beta^{-1} = -\sqrt{5}$, and $\alpha\beta = -1$, this simplifies to F_{2k+1}.

The first five Fibonacci numbers are $F_1 = F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$. After that, we can immediately compute $F_8 = 2F_5F_4 - F_4^2 = 30 - 9 = 21$, $F_9 = F_5^2 + F_4^2 = 25 + 9 = 34$, $F_{16} = 2F_9F_8 - F_8^2 = 987$, $F_{17} = F_9^2 + F_8^2 = 1597$, and $F_{32} = 2F_{17}F_{16} - F_{16}^2 = 2178309$.

(b) As shown in class and in the textbook, $A^k\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}F_{k+1}\\F_k\end{smallmatrix}\right]$. This gives the first column of A^k. Now for any 2×2 maxtrix X, we have that the second column of XA is $X\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, which is the first column of X. Applying this with $X = A^{k-1}$ shows that the second column of A is $\left[\begin{smallmatrix}F_k\\F_{k-1}\end{smallmatrix}\right]$.

Now

$$A^{2k} = (A^k)^2 = \begin{bmatrix} F_{k+1}^2 + F_k^2 & F_{k+1}F_k + F_kF_{k-1} \\ F_{k+1}F_k + F_kF_{k-1} & F_k^2 + F_{k-1}^2 \end{bmatrix}.$$

Looking at the first column, we get

$$\begin{aligned} F_{2k+1} &= F_{k+1}^2 + F_k^2 \\ F_{2k} &= F_{k+1}F_k + F_kF_{k-1}. \end{aligned}$$

The first identity is exactly what we want. For the second one, notice that $F_{k-1} = F_{k+1} - F_k$ by the defining recurrence for the Fibonacci sequence, so

$$F_{2k} = F_k(F_{k+1} + F_{k+1} - F_k) = 2F_{k+1}F_k - F_k^2.$$