Math H54 — Honors Linear Algebra and Differential Equations — Spring, 2004

Prof. Haiman

Review problems for Midterm 2

Midterm 2 is Tuesday, April 13, in class, covering homework assignments 5 through 9. Expect this one to be a little tougher than Midterm 1.

No books, notes, calculators or computers are allowed. Bring scratch paper in case you need more work space than is provided on the exam.

1. Let V be the set of 4×4 matrices with all row- and column-sums equal to zero.

(a) Show that V is a subspace of $M_{4,4}$.

(b) Find $\dim(V)$.

2. Consider the four functions in $\mathcal{C}(\mathbb{R})$:

$$f(x) = \cos^2 x, \quad g(x) = \sin^2 x, \quad h(x) = \cos 2x, \quad j(x) = \sin 2x.$$

Are they linearly independent? Prove it if so; otherwise express one of them as a linear combination of the others.

3. Prove that if $\mathrm{CS}(A) = \mathrm{NS}(A)$, then A is a square matrix of even size.

4. Chapter 3 Review Exercise 27.

5. (a) Find real numbers w, x, y, z such that the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ w & x & y & z \end{bmatrix}$$

is $(\lambda - 1)^4$.

(b) Is the resulting matrix diagonalizable? Why or why not?

6. Find a 3×3 matrix X such that

$$X^2 = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}.$$

How many such matrices X are there?

7. Let $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ be the Fibonacci sequence. Suppose we extend the definition of F_k to negative k by requiring that $F_k = F_{k-1} + F_{k-2}$ hold for all k.

(a) Find a matrix A such that

$$\begin{bmatrix} F_{-k} \\ F_{-(k-1)} \end{bmatrix} = A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for all k.

(b) How is F_{-k} related to F_k?

8. Let $V = \text{span}(\{e^x, xe^x, x^2e^x\})$.

(a) Find the matrix of the linear transformation $D\colon V \to V$ defined by differentiation, with respect to the given basis of V.

(b) Find all functions $f(x) \in V$ that are eigenvectors of D, with their corresponding eigenvalues.

9. (a) Compute the angle between the two vectors $\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}^T$, $\begin{bmatrix}0 & 1 & 2 & 1\end{bmatrix}^T$ in $\mathbb{R}^4$ (with the Euclidean inner product).

(b) Find a unit vector perpendicular to both of the above vectors.

10. Extend

$$\begin{bmatrix}1/4\\1/2\\1/4\\3/4\\1/4\end{bmatrix}, \quad \begin{bmatrix}1/2\\0\\1/2\\-1/2\\1/2\end{bmatrix}$$

to an orthonormal basis of the space spanned by the above two vectors and

$$\begin{bmatrix}1\\2\\2\\2\\3\end{bmatrix}.$$

11. Section 4.3, Ex. 25