Math H54 Honors Linear Algebra and Differential Equations Spring, 2004

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Final Exam Solutions

1. For the matrix A shown below, find a factorization $PA = LU$, where P is a permutation matrix, L is lower unit triangular, and U is in row echelon form.

$$A = \begin{bmatrix} -1 & 1 & -1 & 2 \\ 2 & -2 & 2 & -1 \\ 1 & -1 & 2 & -3 \end{bmatrix}.$$

Solution.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} -1 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

2. Suppose A is a 3×3 matrix such that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -5 \end{bmatrix},$$

and $\mathrm{tr}(A) = 10$. Find $\det(A)$.

Solution. The eigenvalues must be 3, 5 and 2, so $\det(A) = 30$.

3. Suppose A is an $m \times n$ matrix, B is an $n \times k$ matrix, and $AB = \mathbf{0}$. What can you say about how $\mathrm{rank}(A)$ and $\mathrm{rank}(B)$ are related?

Solution. The equation implies $\mathrm{CS}(B) \subseteq \mathrm{NS}(A)$, so $\mathrm{rank}(B) \leq n - \mathrm{rank}(A)$.

4. Let $P_{\leq 5}$ be the vector space of polynomials of degree less than or equal to 5. Let $V \subseteq P_{\leq 5}$ be the subset consisting of polynomials $p(x)$ such that $p(0) = p(1) = 0$.

(a) Show that V is a subspace of $P_{\leq 5}$.

(b) Find the dimension of V and a basis of V.

Solution. (a) Clearly $0 \in V$. If $p(x), q(x) \in V$, then $ap(0) + bq(0) = 0$ and $ap(1) + bq(1) = 0$, so $ap(x) + bq(x) \in V$.

(b) $\dim(V) = 4$. A basis is $\{x^2 - x, x^3 - x^2, x^4 - x^3, x^5 - x^4\}$.

5. (a) Compute the angle between the two vectors $\begin{bmatrix} 2 & 1 & 1 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 2 & 1 & -2 & 1 \end{bmatrix}^T$ in $\mathbb{R}^4$, with the usual Euclidean inner product.

(b) Find an orthonormal basis of the subspace spanned by the two vectors in part (a).

Solution. (a) $\cos\theta = 1/2$, $\theta = \pi/3$. (b) $\{\frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 1 & 1 & 2 \end{bmatrix}^T, \frac{1}{\sqrt{30}} \begin{bmatrix} 2 & 1 & -5 & 0 \end{bmatrix}^T\}$.

6. (a) Solve the initial value problem

$$x'(t) + \frac{1}{t+1}x(t) = \frac{2t+1}{t+1}; \quad x(0) = 3.$$

(b) For what values of t is the solution valid?

Solution. (a) $x(t) = t + \frac{3}{t+1}$. (b) Valid for $t \in (-1, \infty)$.

7. Find all solutions of the differential equation

$$x^{\text{iv}}(t) + 2x''(t) + x(t) = 0.$$

Solution. $x(t) = C_1 \sin t + C_2 \cos t + C_3 t \sin t + C_4 t \cos t$.

8. Consider the system of linear differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & c \\ 1 & -1 \end{bmatrix} x(t).$$

(a) For which values of the constant c does every solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ have the property that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$?

(b) For which values of c does there exist a non-zero solution such that $x_1(t) = 0$ for infinitely many values of t?

Solution. The eigenvalues are $-1 \pm \sqrt{c}$. (a) For $c < 1$ we either have real negative eigenvalues, or complex eigenvalues with negative real part, so $\mathbf{x}(t) \to 0$ as $t \to \infty$. (b) For $c < 0$ we have complex eigenvalues, so the solutions are linear combinations of exponentials times sine and cosine functions. These functions take the value 0 for infinitely many t.

9. Solve the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} \mathbf{x}(t); \quad \mathbf{x}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Solution.

$$\mathbf{x}(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t}.$$

10. Find one solution of the system of linear differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \sin t \\ 0 \end{bmatrix}.$$

Solution.

$$\mathbf{x}(t) = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix} \cos t.$$