Math 55 — Discrete Mathematics — Spring 2003

Quiz 8 Solutions

Prove that if n is a positive integer and $\sqrt{n}$ is rational, then n is a square, *i.e.*, $\sqrt{n}$ is an integer. Hint: show that in the prime factorization of n, every prime must occur to an even power.

Let $\sqrt{n} = a/b$, where a and b are integers and we can assume a and b are relatively prime. Then $n = a^2/b^2$, so

$$a^2 = nb^2.$$

If p is a prime factor of n, then p divides a^2 and therefore p divides a. Since a and b are relatively prime, p does not divide b, so the power of p in the factorization of n is the same as the power of p in the factorization of a^2. But a^2 is a square, so that power is even.

This shows that n is a product of primes to even powers, say $n = p_1^{2e_1} \cdots p_k^{2e_k}$. Then $n = m^2$, where $m = p_1^{e_1} \cdots p_k^{e_k}$.

It is also possible to give a variant of this proof without assuming at the start that a and b are relatively prime. If p is a prime factor of n then since $a^2 = nb^2$, and p occurs to an even power in a^2 and also in b^2, it must occur to an even power in n.