

Math 55 — Discrete Mathematics — Spring 2003

Review Problems for 2nd Midterm

Second midterm exam is Thursday, April 10, usual room, usual time. Try to arrive a few minutes early if possible. No books, notes or calculators may be used. Bring scratch paper.

The exam covers material from Homeworks 6–10, excluding the problems from Homework 10 due on Wednesday. The topics covered are matrices and systems of linear equations, error correcting codes, construction of simple proofs, and proofs by mathematical induction. Not covered are strong induction, recursive definitions and structural induction.

The review problems below are meant to be fairly representative of the subject matter and level of difficulty of the exam.

1. Using a single error correcting Reed-Solomon code over \mathbb{Z}_7 with $m = 4$ message symbols and $n = 6$ code symbols, you receive the word

$$\mathbf{r} = [4 \ 5 \ 2 \ 6 \ 2 \ 6].$$

- (a) Verify that \mathbf{r} contains an error by computing $\Delta^m \mathbf{r}$.
(b) For each of the six unit vectors \mathbf{e} with entries in \mathbb{Z}_7 , compute $\Delta^m \mathbf{e}$. (A unit vector has one entry equal to 1 and the rest zero.)
(c) Locate and correct the error in \mathbf{r} by finding a multiple of one of the unit vectors that can be added to \mathbf{r} to get a code vector.

[This method is called error correction by *syndromes*. The computation in part (b) is the construction of a *syndrome table* for the possible error positions. Syndrome correction is most efficient for codes that correct a small number of errors, *e.g.* one or two.]

2. (a) Write down the code matrix for the Reed-Solomon code over \mathbb{Z}_{11} with 3 message symbols and 9 code symbols.
(b) Encode the message vector $[4 \ 8 \ 3]$.
(c) How many errors can this code correct?
(d) Is this code better than the triple redundancy code with matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}?$$

Why or why not?

3. Using a two-error correcting Reed-Solomon code over \mathbb{Z}_{11} with $m = 4$ and $n = 8$, you receive the vector

$$[3 \ 3 \ 1 \ 6 \ 7 \ 3 \ 1 \ 4].$$

- (a) Find an “error locator” polynomial $E(t)$ of degree ≤ 2 and not identically zero, such that there exists $Q(t)$ of degree ≤ 5 solving the key equation

$$Q(i) = r_i E(i), \quad i = 0, 1, \dots, 7.$$

- (b) Locate the possible error positions (if any) by evaluating $E(i)$ for each i .

Note that this problem does not ask you to find $Q(t)$ or the original message, only the possible error positions.

4. The code matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 1 & 0 \\ 1 & 2 & 3 & 0 & 1 & 4 \\ 2 & 3 & 0 & 1 & 1 & 2 \\ 3 & 0 & 1 & 2 & 1 & 4 \end{bmatrix}$$

defines a single error correcting linear code over \mathbb{Z}_5 with 4 message symbols and 6 code symbols. Using this code, you receive the vector

$$[2 \ 3 \ 0 \ 0 \ 0 \ 3].$$

Verify that this received vector has no errors by finding the message vector that corresponds to it.

5. Prove that if n is an odd integer, then $n^4 - 1$ is divisible by 16.

6. Prove that there are infinitely many primes congruent to 2 (mod 3).

7. Prove that the product of any four consecutive integers is divisible by 24.

8. Prove that $2^n < n!$ for all integers $n > 3$.

9. Prove that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for every positive integer n .

10. If \mathbf{A} is the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, prove that

$$\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

for all positive integers n .