Answers to review problems for 1st midterm

1. (a) $f(n) = O(n^2 2^n)$ (b) $f(n) = O(n^6)$.

2. (a) The inner loop sets *repeat* to 1 if a_i is a repetition of an earlier element. The *total* gets incremented once for each non-repeated element, so at the end it counts the distinct elements.

(b) The inner loop takes $O(n)$ steps, and the outer loop performs the inner one $O(n)$ times, for a complexity of $O(n^2)$.

(c) If the list is pre-sorted, we can check if a_i is a repeat by just comparing it to a_{i-1}. The counting phase takes $O(n)$ steps, so the complexity is dominated by the inital sorting phase, giving $O(n \log n)$ overall.

3. $\gcd(8918, 1001) = 91$, $\operatorname{lcm}(8919, 1001) = 98098$.

4. $\gcd(2^3 3^4 5^7 7, 2^7 3^5 5^3 11) = 2^3 3^4 5^3$, $\operatorname{lcm}(2^3 3^4 5^7 7, 2^7 3^5 5^3 11) = 2^7 3^5 5^7 \cdot 7 \cdot 11$.

5. If $k^2 | n$ and $k \neq 1$, let p be a prime factor of k. Then $p^2 | n$. Since prime factorization is unique, this contradicts the assumption that n is a product of distinct primes.

6. $2^2 3^4 5^6 7^{10} = (2 \cdot 3^2 5^3 7^5)^2$.

7. Since $10 \equiv -4$ and $15 \equiv 29 \pmod 7$, the desired identity follows immediately.

8. $\gcd(84, 119) = 7 = 10 \cdot 84 - 7 \cdot 119$

9. (a) 11 (b) $x = 10$.

10. (a) Passes Fermat (b) Fails Miller; hence is composite.

11. $x \equiv 17 \pmod{140}$.

12. $d = 343$.

13. $2821 = 7 \cdot 11 \cdot 31$. Since $7-1 = 6$, $11-1 = 10$ and $31-1 = 30$ all divide 2820, Fermat's theorem shows that $x^{2820} \equiv 1 \pmod 7$, $\pmod{11}$ and $\pmod{31}$ for all x relatively prime to 2821. By the Chinese Remainder Theorem, this implies $x^{2820} \equiv 1 \pmod{2821}$.

14. One correct solution is $m = 1$, $n = p = 2$. A less silly solution, with the three numbers distinct and none of them equal to 1, is $m = 2$, $n = 3$, $p = 4$. If you demand that no two of them are relatively prime (although the problem does not require this), a solution is $m = 6$, $n = 10$, $p = 15$.

15. $\mathbf{A}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}_2$. From this we see that $\mathbf{A}^{n+2} = -\mathbf{A}^n$ for all n, leading to the rule

$$\mathbf{A}^n = \begin{cases} \mathbf{A} & \text{if } n \equiv 1 \pmod 4, \\ -\mathbf{I}_2 & \text{if } n \equiv 2 \pmod 4, \\ -\mathbf{A} & \text{if } n \equiv 3 \pmod 4, \\ \mathbf{I}_2 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$