Homework 9 Solutions

2: If n is odd then $n \equiv 1$, 3, 5 or 7 $\pmod 8$. In each case we have $n^2 \equiv 1 \pmod 8$. An alternative proof is to write $n = 2k+1$, so $n^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. The number $k(k+1)$ is even, no matter whether k is even or odd, so 8 divides $4k(k+1)$, and hence $n^2 \equiv 1 \pmod 8$.

10: Following the hint, let

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2.$$

Then calculate

$$\begin{aligned} x^2 + y^2 &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= (m^2 + n^2)^2 \\ &= z^2. \end{aligned}$$

There are infinitely many choices for m and n, giving infinitely many integer solutions of $x^2 + y^2 = z^2$.

10 Extra: Trying small values in the recipe for solutions above, we soon find that $m = 1$, $n = 4$ gives $(8 : 15 : 17)$. The problem doesn't ask for a proof that this is the smallest possibility, but you could prove it by checking all possibilities with hypotenuse length less than 17.

25 Extra: Given consecutive odd integers p, $p+2$, $p+4$, observe that if $p \equiv 0 \pmod 3$ then $3|p$, if $p \equiv 1 \pmod 3$ then $3|p+2$, and if $p \equiv 2 \pmod 3$, then $3|p+4$. In every case, one of the three numbers is divisible by 3, so they can't all be prime unless $p = 3$.

30: We might suppose that $p_1, \ldots, p_l$ is a list of all the primes $\equiv 1 \pmod 4$, and construct $Q = 4p_1 \cdots p_l + 1$, so $Q \equiv 1 \pmod 4$ and no p_i divides Q. Then letting $Q = q_1 \cdots q_t$ be the prime factorization, we might hope to show that some $q_i \equiv 1 \pmod 4$, yielding a contradiction. But this last conclusion doesn't follow, since if t is even and every $q_i \equiv 3 \pmod 4$, we would still have $Q \equiv 1 \pmod 4$.