Homework 8 Solutions

6. Extending the difference table from the bottom gives

$$\begin{array}{ccccccccc} 1 & 1 & 2 & 3 & 3 & 1 & -4 & -13 & -27 \\ 0 & 1 & 1 & 0 & -2 & -5 & -9 & -14 & \\ 1 & 0 & -1 & -2 & -3 & -4 & -5 & & \\ -1 & -1 & -1 & -1 & -1 & -1 & & & \end{array}$$

7. From the left column of the above difference table we get

$$f(x) = C_0(x) + C_2(x) - C_3(x) = 1 - \frac{5\,x}{6} + x^2 - \frac{x^3}{6}$$

13. Make difference tables (mod 11) for the sequences R_i and iR_i:

$$\begin{array}{ccccc} 9 & 2 & 9 & 1 & 7 \\ 4 & 7 & 3 & 6 & \\ 3 & 7 & 3 & & \\ 4 & 7 & & & \\ 3 & & & & \end{array} \quad , \quad \begin{array}{ccccc} 0 & 2 & 7 & 3 & 6 \\ 2 & 5 & 7 & 3 & \\ 3 & 2 & 7 & & \\ 10 & 5 & & & \\ 6 & & & & \end{array}$$

From the last rows, the coefficients of $E(t) = v_0 + v_1 t$ satisfy

$$\begin{bmatrix} 3 & 6 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \mathbf{0}.$$

For a solution take $v_1 = 1$, so $v_0 = 9$, and $E(t) = t + 9$. To find $Q(t)$, make a difference table for $Q(i) = R_i E(i)$:

$$\begin{array}{ccccc} 4 & 9 & 0 & 1 & 3 \\ 5 & 2 & 1 & 2 & \\ 8 & 10 & 1 & & \\ 2 & 2 & & & \end{array} .$$

Then $Q(t) = 4C_0(t) + 5C_1(t) + 8C_2(t) + 2C_3(t) = 4 + 9\,t + 3\,t^2 + 4\,t^3$, and $P(t) = Q(t)/E(t) = 9 + 4\,t^2$. The message was therefore $\begin{bmatrix} 9 & 0 & 4 \end{bmatrix}$. To check this, encode the message:

$$\begin{bmatrix} 9 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 9 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 3 & 1 & 7 \end{bmatrix}.$$

This differs from the received vector by an error in the middle position ($i = 2$).

To do the above problem using the key equation directly we put $E(t) = v_0 + v_1 t$ and $Q(t) = u_0 + u_1 t + u_2 t^2 + u_3 t^3$. Then setting $Q(i) = R_i E(i)$ for $i = 0, 1, \ldots, 4$ gives the system of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 9 & 9 \\ 1 & 2 & 4 & 8 & 2 & 4 \\ 1 & 3 & 9 & 5 & 10 & 8 \\ 1 & 4 & 5 & 9 & 4 & 5 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ v_0 \\ v_1 \end{bmatrix} = \mathbf{0}.$$

Row-reducing $\pmod{11}$ gives the equivalent system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ v_0 \\ v_1 \end{bmatrix} = \mathbf{0},$$

with a solution $\begin{bmatrix} u_0 & u_1 & u_2 & u_3 & v_0 & v_1 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 3 & 4 & 9 & 1 \end{bmatrix}$, so $Q(t) = 4 + 9t + 3t^2 + 4t^3$ and $E(t) = 9 + t$. The rest is the same as in the other solution.

14. To handle an error rate of $1/3$ means we need $e = n/3$. Since $n = m + 2e$ for a Reed-Solomon code, that implies $m = n/3$. Hence the data rate for a Reed-Solomon code that can correct $n/3$ errors in $1/3$. A triple redundancy code with $n = 3$ and $m = 1$ corrects one error in three symbols, so it also has a data rate of $1/3$ and tolerates an error rate of $1/3$.

The difference is that the Reed-Solomon code can have more than $n = 3$ symbols in one code block. For example, a Reed-Solomon code of length 150 can correct errors in any 50 of the 150 symbols. A triple-redundancy code with 50 blocks of length 3 carries the same number of message symbols (50), but it can correct 50 errors only if they happen to distribute themselves evenly with one error in each group of 3 symbols, an unlikely possibility.