Homework 15 Solutions

1. Chapter 8.1, #11(c). See text.

2. The graph in the previous problem is an example of an *interval graph*: the intersection graph of a collection of intervals in $\mathbb{R}$. Show that an interval graph G cannot contain a 4-cycle without diagonals. In other words, if $\{w, x, y, z\}$ are the vertices of a 4-cycle in an interval graph, then G contains at least 5 of the 6 possible edges between these vertices.

Suppose we had four intervals $A_1 = [a_1, b_1]$, $A_2 = [a_2, b_2]$, $A_3 = [a_3, b_3]$, $A_4 = [a_4, b_4]$ whose intersection graph was a 4-cycle, with edges $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_3, A_4\}$, $\{A_4, A_1\}$, and no other edges. Then $A_1 \cap A_3 = \emptyset$. After switching A_1 and A_3 if necessary (which is OK, because it's an isomorphism from the 4-cycle to itself), we can assume that $b_1 < a_3$. Since A_2 overlaps both A_1 and A_3, we must have $a_2 \leq b_1$ and $b_2 \geq a_3$. Similarly, we must have $a_4 \leq b_1$ and $b_4 \geq a_3$. But then A_2 overlaps A_4, since both contain a_3. This is a contradiction.

3. Chapter 8.3, #43, #44. A possible isomorphism for #43 is

$$u_1 \to v_1,\ u_2 \to v_9,\ u_3 \to v_4,\ u_4 \to v_5,\ u_5 \to v_6$$
$$u_6 \to v_7,\ u_7 \to v_8,\ u_8 \to v_3,\ u_9 \to v_{10},\ u_{10} \to v_2.$$

The graphs in #44 are not isomorphic. One way to prove this is to observe that the complement of the first graph consists of two disjoint 4-cycles ($u_1u_3u_5u_7$ and $u_2u_4u_6u_8$), while the complement of the second graph is an 8-cycle ($v_1v_4v_7v_2v_5v_8v_3v_6$).

4. Chapter 8.6, #4, #5. See text for the path. Another path that works is a, c, f, i, m, p, s, z. The length is 16.

5. In class we showed that the Ramsey number $R(4, 4)$ is less than or equal to 18. In this exercise, we will prove that $R(4, 4) = 18$ by constructing a 2-coloring of the edges of K_{17} such that there is no red K_4 and no white K_4.

Our K_{17} will have vertex set $V = \{0, 1, 2, \ldots, 16\} = \mathbb{Z}_{17}$. Color it by the rule that an edge $\{x, y\}$ is

$$\text{red if } y - x \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$$
$$\text{white if } y - x \in \{\pm 3, \pm 5, \pm 6, \pm 7\},$$

where all arithmetic is $\pmod{17}$.

(a) Show that the function $f(x) = 3x$ is an isomorphism from K_{17} to itself that sends red edges to white edges and white edges to red edges. Therefore our coloring contains a red K_4 if and only if it contains a white K_4, so it is enough to prove it contains no red K_4.

Since $3 \cdot 6 \equiv 1 \pmod{17}$, the function $g(y) = 6y$ is inverse to $f(x) = 3x$. Therefore f is one-to-one and onto, and since the graph is complete, it's an isomorphism. To show that f reverses colors, note that $f(y) - f(x) = 3(y - x)$, so we just have to check that multiplication by 3 (mod 17) sends the members of each set $R = \{\pm 1, \pm 2, \pm 4, \pm 8\}$ and $W = \{\pm 3, \pm 5, \pm 6, \pm 7\}$ to the other. This is true, since $3 \cdot 1 \equiv 3$, $3 \cdot 2 \equiv 6$, $3 \cdot 4 \equiv -5$, and $3 \cdot 8 \equiv 7$.

(b) Show that if w, x, y, z are the vertices of a red K_4, then so are $w + a, x + a, y + a, z + a$. By taking $a = -w$, show that if there is a red K_4 then there is one that contains vertex 0.

The color of an edge is determined by the difference of its vertices, and $(x+a)-(y+a) = x-y$, so $\{x+a, y+a\}$ is the same color as $\{x, y\}$ for every a. Hence if w, x, y, z is a red K_4, then so is $0, x-w, y-w, z-w$.

(c) Show that if $0, x, y, z$ are the vertices of a red K_4, then so are $0, ax, ay, az$ for any $a \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. Also show that if $a \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$, then $a^{-1} \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. By taking $a = x^{-1}$, show that if there is a red K_4, then there is one that contains vertices 0 and 1.

Since $ax - ay = a(x-y)$, we see that $\{ax, ay\}$ is the same color as $\{x, y\}$ if multiplication by a sends the members of each set R and W into the same set. You could check this for every $a \in R$. Or, more cleverly, observe that $2^4 = 16 \equiv -1 \pmod{17}$, so R is the set of all powers of 2 (mod 17). From this description it is clear that if a and x are both in R then so is ax. Moreover, if $a \in R$, since multiplication by a is a bijection from $R \cup W$ to itself, and it sends R into R, it must send W into W.

Similarly, you can just check that the inverse of every $a \in R$ is in R, or deduce this from the fact that R is the set of all powers of 2.

It follows that if $0, x, y, z$ is a red K_4, so x is in R, then $0, 1, x^{-1}y, x^{-1}z$ is a red K_4.

(d) Find all the vertices $x \neq 0, 1$ such that both edges $\{0, x\}$ and $\{1, x\}$ are red.

For $\{0, x\}$ to be red we must have $x \in R = \{\pm 1, \pm 2, \pm 4, \pm 8\} = \{1, 2, 4, 8, 9, 13, 15, 16\}$. For $\{1, x\}$ to be red we must have $x - 1 \in R$, so $x \in \{2, 3, 5, 9, 10, 14, 16, 0\}$. The intersection of these two sets is $\{2, 9, 16\}$, so these are the only possibilities for x.

(e) Prove that there is no red K_4, and therefore also no white K_4, in this coloring of K_{17}.

If there is a red K_4 then parts (b)–(d) show that there is one of the form $\{0, 1, x, y\}$ with $x, y \in \{2, 9, 16\}$. But all three edges $\{2, 9\}$, $\{2, 16\}$ and $\{9, 16\}$ are white, since the differences for these edges are ± 7 and ± 3. Hence there is no red K_4. By part (a), there is also no white K_4.

This example shows that the Ramsey number $R(4, 4)$ is greater than 17. In the lecture we showed that $R(4, 4) \leq 18$, so it follows that $R(4, 4) = 18$.