## Homework 13 Solutions

4.1 #18: The number of integers in $\{1,\ldots,999\}$ divisible by $d$ is $\lfloor 999/d \rfloor$ for any $d$. The ones divisible by both 7 and 11 are the ones divisible by 77, since 7 and 11 are relatively prime. Using these facts, we get (a) $\lfloor 999/7 \rfloor$, (b) $\lfloor 999/7 \rfloor - \lfloor 999/77 \rfloor$, (c) $\lfloor 999/77 \rfloor$, (d) $\lfloor 999/7 \rfloor + \lfloor 999/11 \rfloor - \lfloor 999/77 \rfloor$, (e) $\lfloor 999/7 \rfloor + \lfloor 999/11 \rfloor - 2\lfloor 999/77 \rfloor$, (f) $999 - \lfloor 999/7 \rfloor - \lfloor 999/11 \rfloor + \lfloor 999/77 \rfloor$.

The other two parts are different: (g) $9+9\cdot 9+9\cdot 9\cdot 8$, (f) $4+9+4\cdot 8+9\cdot 8+4\cdot 8+4\cdot 8\cdot 7$ or equally $9+9\cdot 9+9\cdot 9\cdot 8-5-5\cdot 8-5\cdot 8\cdot 8$. The first answer is by counting cases (one digit $x$)+(two digits $x0$)+(two digits, no zero)+(three digits $xy0$)+(three digits $x0y$)+(three digits, no zero). The second is by subtracting the number of *odd* numbers from 1 to 999 with distinct digits from the total in (g).

4.1 #46: $53 \cdot 63^7$

4.1 #50: 70 ways. A cleverer way to get this answer is to note that the winning team in the series always wins exactly four games. Then there are 2 choices for the winning team, $\binom{7}{4}$ choices for which 4 games they win, for a total of $70 = 2\binom{7}{4}$.

4.3 #30: (a) $\binom{16}{5} - \binom{9}{5}$ (all possible committees minus men-only committees) (b) $\binom{16}{5} - \binom{9}{5} - \binom{7}{5}$.

4.5 #14: $\binom{4+17-1}{17} = \binom{20}{17}$.

4.5 #30: $\binom{11}{1,4,4,2}$

Extra problem: Since the code is $e$-error correcting, every received word has distance $\leq e$ from at most one codeword, and perfection means every received word has distance $\leq e$ from exactly one codeword. The number of words at distance $k$ from a given codeword $w$ is $\binom{n}{k}(p-1)^k$, since there are $\binom{n}{k}$ ways to choose $k$ error positions, and in each position $i$ there are $p-1$ choices for a symbol different from the symbol $w_i$. The number of words at distance $\leq e$ from each code word $w$ is therefore

$$1 + \binom{n}{1}(p-1) + \cdots + \binom{n}{e}(p-1)^e.$$

The total number of words $p^n$ is equal to the number of codewords times the formula above, so the number given by the formula must divide $p^n$.

Computing for $p = 2$ and all $n < 25$ with a calculator, you should find that there are two possible values of $n$: $n = 7$ and $n = 23$. The obvious perfect 3-error correcting code with $n = 7$ is the seven-fold redundancy code, not a very interesting example.

The case $n = 23$ is interesting. Our computation doesn't prove that a perfect 3-error correcting code of this length actually exists, but in fact one does. It is called the *Golay code*, and it has many remarkable mathematical properties.