Homework 11 Solutions

3.3 #34: *Theorem:* It takes $n-1$ breaks to separate the chocolate bar into single squares, no matter what choices are made.

Proof: We use strong induction on n. If $n = 1$, the chocolate bar is already a single square, and the number of breaks required is $n - 1 = 0$.

If $n > 1$, then after the first break we will have two bars, say with r and s squares respectively. Since r and s are less than n, we can assume by induction that it takes $r - 1$ breaks to separate the bar with r squares and $s-1$ breaks to separate the bar with s squares. Counting the first break, we need $(r-1)+(s-1)+1$ breaks in all, and $(r-1)+(s-1)+1 = r + s - 1 = n - 1$.

3.3 #40: Proof by strong induction on n. If $n = 1$, we don't split the pile at all, and get a total of $n(n-1)/2 = 0$. If $n > 1$ and we split the pile into r and s stones, then we can assume by induction that on further splitting the r stone pile contributes $r(r-1)/2$ to the total and the s stone pile contributes $s(s-1)/2$. Adding rs for the first split gives a total of

$$rs + r(r-1)/2 + s(s-1)/2 = (r^2 + 2rs + s^2 - r - s)/2 = (r+s)(r+s-1)/2.$$

Sinc $r + s = n$, this is equal to $n(n-1)/2$.

3.4 #12: Basis step: for $n = 1$, both sides reduce to 1.

Induction step: assume

$$f_1^2 + \cdots + f_n^2 = f_n f_{n+1}$$

and add f_{n+1}^2 to both sides, to get

$$f_1^2 + \cdots + f_n^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = (f_n + f_{n+1})f_{n+1} = f_{n+1}f_{n+2}.$$

#18: Basis step $n = 1$ follows since $f_0 = 0$ and $f_1 = f_2 = 1$.

For the induction step, assume

$$\mathbf{A}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

and multiply both sides by $\mathbf{A}$ to get

$$\mathbf{A}^{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_n + f_{n+1} & f_{n-1} + f_n \\ f_{n+1} & f_n \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}.$$