

Homework 15

1. Chapter 8.1, #11(c).

2. The graph in the previous problem is an example of an *interval graph*: the intersection graph of a collection of intervals in \mathbb{R} . Show that an interval graph G cannot contain a 4-cycle without diagonals. In other words, if $\{w, x, y, z\}$ are the vertices of a 4-cycle in an interval graph, then G contains at least 5 of the 6 possible edges between these vertices.

3. Chapter 8.3, #43, #44

4. Chapter 8.6, #4, #5

5. In class we showed that the Ramsey number $R(4, 4)$ is less than or equal to 18. In this exercise, we will prove that $R(4, 4) = 18$ by constructing a 2-coloring of the edges of K_{17} such that there is no red K_4 and no white K_4 .

Our K_{17} will have vertex set $V = \{0, 1, 2, \dots, 16\} = \mathbb{Z}_{17}$. Color it by the rule that an edge $\{x, y\}$ is

$$\begin{aligned} &\text{red if } y - x \in \{\pm 1, \pm 2, \pm 4, \pm 8\} \\ &\text{white if } y - x \in \{\pm 3, \pm 5, \pm 6, \pm 7\}, \end{aligned}$$

where all arithmetic is (mod 17).

(a) Show that the function $f(x) = 3x$ is an isomorphism from K_{17} to itself that sends red edges to white edges and white edges to red edges. Therefore our coloring contains a red K_4 if and only if it contains a white K_4 , so it is enough to prove it contains no red K_4 .

(b) Show that if w, x, y, z are the vertices of a red K_4 , then so are $w + a, x + a, y + a, z + a$. By taking $a = -w$, show that if there is a red K_4 then there is one that contains vertex 0.

(c) Show that if $0, x, y, z$ are the vertices of a red K_4 , then so are $0, ax, ay, az$ for any $a \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. Also show that if $a \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$, then $a^{-1} \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. By taking $a = x^{-1}$, show that if there is a red K_4 , then there is one that contains vertices 0 and 1.

(d) Find all the vertices $x \neq 0, 1$ such that both edges $\{0, x\}$ and $\{1, x\}$ are red.

(e) Prove that there is no red K_4 , and therefore also no white K_4 , in this coloring of K_{17} .