Math 55: Discrete Mathematics, Fall 2008
Second Midterm Exam Solutions

1. Let (a_n) be the sequence recursively defined by $a_0 = 0$, $a_{n+1} = a_n + 2n + 1$ for $n \geq 0$. Find an exact formula for a_n and prove your formula is correct.

The formula is $a_n = n^2$. To prove it, use induction. The basis step $a_0 = 0$ is given. For the induction step, if $n \geq 0$ and $a_n = n^2$, then $a_{n+1} = n^2 + 2n + 1 = (n+1)^2$.

2. (a) 15 guests at a party draw one gift each from a basket containing 20 different gifts. How many distributions of the gifts are possible?

(b) What if the gift basket contains 10 identical whistles and 10 identical balloons, and we do not distinguish between identical gifts?

(a) $P(20, 15) = 20 \cdot 19 \cdot 18 \cdots 6$

(b) If the guests draw k whistles and $15 - k$ balloons, these may be distributed in $\binom{15}{k}$ ways. Since we must have $5 \leq k \leq 10$, the total number of ways is

$$\binom{15}{5} + \binom{15}{6} + \binom{15}{7} + \binom{15}{8} + \binom{15}{9} + \binom{15}{10}.$$

3. If the letters in the word BLUBBERER are scrambled at random (all rearrangements having equal probability), what is the probability that the two R's are adjacent to each other in the scrambled word?

Treat the pair of R's as a single letter to count rearrangements in which they are adjacent. The resulting probability is

$$\binom{8}{3,2,1,1,1} \Big/ \binom{9}{3,2,2,1,1} = 2/9.$$

4. You have two standard decks of 52 playing cards. One deck has been sorted so the top card is the ace of spades. The other deck has been shuffled randomly. You don't know which deck is which. If you choose one deck at random, look at the top card, and see the ace of spades, what is the probability that you have chosen the sorted deck?

Let E be the event that you choose the sorted deck, F the event that you draw the ace of spades. Then $p(E) = 1/2$, $p(F|E) = 1$, $p(F|\overline{E}) = 1/52$. By Bayes' Theorem,

$$p(E|F) = 1(1/2)/(1(1/2) + (1/52)(1/2)) = 52/53.$$

5. (a) Find the probability that 6 comes up k times on n rolls of a fair six-sided die.

(b) Suppose I offer to pay 2^k dollars if 6 comes up k times. What is the expected payoff on n rolls of the die? Express your answer in the simplest possible form.

(a) $\binom{n}{k}(1/6)^k(5/6)^{n-k}$.

(b) $\sum_{k=0}^{n} \binom{n}{k} 2^k (1/6)^k (5/6)^{n-k} = (2/6 + 5/6)^n = (7/6)^n$, by the binomial theorem.