SPRING HOMEWORK PROBLEMS, SET 7

1. Let k be an algebraically closed field, $M_{r,s}(k)$ the set of $r \times s$ matrices over k, which we identify with the set of closed points of $\mathbb{A}_k^{rs}$. Let $X \subseteq \mathbb{A}_k^{rs+st}$ be the variety whose closed points are pairs $A \in M_{r,s}(k)$, $B \in M_{s,t}(k)$ such that $AB = 0$. Determine the irreducible components of X and their dimensions, for all r, s, t.

2. Let $C = V(y^2 - x^2(x+1))$ be the *nodal cubic curve* in $\mathbb{A}_k^2$ (k an algebraically closed field).

(a) Construct a finite, surjective morphism $g\colon \mathbb{A}_k^1 \to C$ which is one-to-one except over the origin, whose fiber has two points.

(b) Let $f\colon \mathbb{A}^2 \to \mathbb{A}^1 \times C$ be the base extension by $\mathbb{A}^1$ of the morphism g above. Show that there exists a closed point $p \in \mathbb{A}^2$ and an irreducible curve $Z \subseteq \mathbb{A}^1 \times C$ containing $f(p)$, such that no irreducible curve in $\mathbb{A}^2$ containing p maps onto Z.

(c) Verify that (b) provides a counterexample to the "going-up" theorem in the absence of the hypothesis that the smaller domain $A \subseteq B$ is normal.

3. (a) Show that if $\mathcal{A}$ is the category of $\mathcal{O}_X$-modules, then $\mathcal{H}om^\bullet(\mathcal{O}_X[-n], B) = B[n]$ for any complex B in $\mathbf{C}(\mathcal{A})$ (cf. Derived Category notes, Remark 1.8).

(b) For any $\mathcal{A}$, show that $\operatorname{Hom}^\bullet(A, B[n]) = \operatorname{Hom}^\bullet(A, B)[n]$. Deduce that $H^n(\operatorname{Hom}^\bullet(A, B)) = \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A, B[n])$.

4. (a) Prove the following more general form of Derived Category notes, Prop. 1.9(b,c): A composite homomorphism gf in $\mathbf{C}(\mathcal{A})$ is null-homotopic if and only g factors through $C(f)$, if and only if f factors through $C(g)[-1]$.

(b) Use (a) to give an alternate proof of Derived Category notes, Corollary 3.14.

(c) Give a third proof of Corollary 3.14 by showing that it holds in any additive category with shift functors and distinguished exact triangles satisfying axioms (o–iv) in Proposition 2.3, hence in any triangulated category. In particular, the counterpart of Corollary 3.14 holds in $\mathbf{D}(\mathcal{A})$.

(d) Show that if a morphism of triangles is an isomorphism at two vertices, then it is an isomorphism at all three, in any triangulated category.

5. Show that the triangle in the derived category notes, Prop. 3.9, is also isomorphic to a rotation of the triangle based on $g\colon B \to C$.

6. Given a commutative diagram in $\mathbf{C}(\mathcal{A})$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns, show that the square formed by connecting homomorphisms in $\mathbf{D}(\mathcal{A})$ anticommutes:

$$\begin{tikzcd}
C'' \arrow[r] \arrow[d] & A''[1] \arrow[d] \\
C[1] \arrow[r] & A[2]
\end{tikzcd} .$$