Spring homework problems, set 2

1. Let $R = k[y_0, y_1, y_2 \ldots]$ be a polynomial ring in infinitely many variables, $J \subseteq R$ the ideal generated by the homogeneous quadratic polynomials $y_i y_j - y_0 y_{i+j}$ for all i, j, and $S = R/J$, a graded k-algebra generated by S_1, but not finitely generated. Using EGA (II: 2.4.8), prove that $\mathrm{Proj}(S) \cong \mathbb{A}^1_k = \mathrm{Spec}\, k[t]$; and hence there is a closed immersion of $\mathbb{A}^1_k$ into $\mathbb{P}^\infty_k = \mathrm{Proj}(R)$.

[Note that, at least if k is an algebraically closed field, it is reasonable to regard $\mathrm{Proj}(R)$ as an infinite dimensional projective space, and the above immersion as being given in coordinates by $t \mapsto (1 : t : t^2 : \cdots)$. The corresponding immersion $t \mapsto (1 : t : \cdots : t^{N-1})$ of $\mathbb{A}^1$ into $\mathbb{P}^N$ is not closed; in fact it extends to a closed immersion $\mathbb{P}^1 \to \mathbb{P}^N$ given by $(s : t) \mapsto (s^N : s^{N-1}t : \cdots : t^N)$, mapping the point $(0 : 1)$ (at "$t = \infty$") to $(0 : 0 : \cdots : 1)$. Such an extension is of course not possible for the immersion into $\mathbb{P}^\infty$. Later we will see that $\mathbb{A}^1_k$ is not isomorphic to any closed subscheme of $\mathbb{P}^N_k$ (for finite N).]

2. Let $F_{n,\mathbb{C}}$ be the "Fermat curve" in $\mathbb{P}^2_{\mathbb{C}}$, defined by the equation $x^n + y^n = z^n$. More precisely, this means $F_{n,\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x, y, z]/(x^n + y^n - z^n))$. Note that $F_{n,\mathbb{C}} = F_{n,\mathbb{Z}} \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{C})$, where $F_{n,\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[x, y, z]/(x^n + y^n - z^n))$, so the $\mathbb{C}$-points of $F_{n,\mathbb{Z}}$ are identified with the closed points of $F_{n,\mathbb{C}}$. Show that the statement *for every odd prime n, $F_{n,\mathbb{Z}}$ has exactly three $\mathbb{Q}$-points* is equivalent to a famous theorem.

3. Let $X = \mathbb{P}^N_k = \mathrm{Proj}(k[x_0, \ldots, x_N])$. Let $f \colon \mathcal{O}_X^{N+1} \to \mathcal{O}_X(1)$ be the $\mathcal{O}_X$-module homomorphism that maps the i-th unit coordinate section e_i of $\mathcal{O}_X$ to the global section x_i of $\mathcal{O}_X(1)$. Then f induces an $\mathcal{O}_X$-algebra homomorphism $\mathcal{O}_X[t_0, \ldots, t_N] = \mathbf{S}(\mathcal{O}^{N+1}) \to \mathbf{S}(\mathcal{O}(1))$.

(a) Show that the corresponding morphism from $L = \mathbf{V}(\mathcal{O}(1))$ to $E = \mathbb{A}^{N+1} \times \mathbb{P}^N = \mathbf{V}(\mathcal{O}^{N+1})$ is a closed immersion of L into the trivial vector bundle E as a rank-1 sub-bundle.

(b) Let $\mathcal{Q}^\vee = \ker(f)$. Show that $\mathcal{Q}^\vee$ is locally free of rank N, and $Q = \mathbf{V}(\mathcal{Q}^\vee)$ can be identified with E/L as a vector bundle (we call L the *tautological line bundle* and Q the *tautological quotient bundle*).

(c) Let $Y = \mathbb{P}(\mathcal{Q}^\vee) = \mathrm{Proj}(\mathbf{S}(\mathcal{Q}^\vee))$, a scheme projective over $X = \mathbb{P}^N_k$. When k is an algebraically closed field, show that the k-points of Y are naturally identified with pairs $L_1 \subseteq L_2 \subseteq k^{N+1}$, where L_1 and L_2 are linear subspaces of dimensions 1 and 2, respectively. (This can be repeated to give the space of "flags" $L_1 \subseteq L_2 \subseteq \cdots L_r \subseteq k^{N+1}$ the structure of an iterated projective space bundle.)

4. Let p be a k-point of the projective line $X = \mathbb{P}^1$ over a field k. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the ideal sheaf such that $V(\mathcal{I}) = \{p\}$ with the induced structure of reduced closed subscheme. Show that $\mathcal{I}$ is an invertible sheaf of $\mathcal{O}_X$-modules; in fact, find n such that $\mathcal{I} \cong \mathcal{O}(n)$. [Note that since every global function on $\mathbb{P}^1$ is constant, the only global section of $\mathcal{I}$ is the zero section, so n should be negative.]

5. Let $E \subseteq \mathbb{P}^2_k$ be the projective elliptic curve with equation $y^2 z = x(x^2 - z^2)$ (the projective closure of affine plane curve $y^2 = x(x^2 - 1)$), over a field k. (a) Show that if p is any k-point of E, its ideal sheaf $\mathcal{I}$ (as in problem 4) is invertible, and that there exists a

point p such that if $\mathcal{O}_E(1)$ is the twisting sheaf induced by the given projective embedding, we have $\mathcal{I}^{\otimes 3} \cong \mathcal{O}_E(-1)$. (b) Show that $\Gamma(E, \mathcal{O}_E(1))$ is a 3-dimensional vector space over k. (c) Deduce that E cannot be isomorphic to $\mathbb{P}^1(k)$.

A hint for (b): $\mathcal{O}_E(1)$ is trivial on $E \cap D_+(z) = E \setminus \{q\}$ where q is the point "at infinity." Thus global sections of $\mathcal{O}_E(1)$ are given by polynomials in the functions x and y which have no pole when rewritten in local coordinates on a neighborhood of q. Later we'll see a way to solve (b) instantly, using sheaf cohomology.