Homework problems, Set 3

1. Prove (EGA 0:2.2.5).

2. Let $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ be a polynomial ring in infinitely many variables over a field k, modulo the ideal generated by the squares of the variables. Show that A has a unique prime ideal $\mathfrak{m} = (x_1, x_2, \ldots)$, but A is not Noetherian. In particular, Spec(A) being a Noetherian space does not imply that A is a Noetherian ring.

3. Mumford's method of constructing rings and modules of fractions is to define $S^{-1}A = A[u_s : s \in S]/(su_s - 1 : s \in S)$ and $S^{-1}M = S^{-1}A \otimes_A M$. (a) Deduce the universal properties directly from this definition and the properties of polynomial rings, quotient rings, and tensor products. (b) Defining $a/s = u_s a \in S^{-1}A$ and $m/s = u_s m \in S^{-1}M$, verify that these symbols add and multiply according to the usual rules for fractions, and that set of all elements a/s (resp. m/s) is a subring of $S^{-1}A$ (resp. submodule of $S^{-1}M$), containing a generating set; hence every element has this form.

To complete the explicit description of $S^{-1}M$ we need to know when $m_1/s_1 = m_2/s_2$, which is equivalent to determining when $(m_1 s_2 - m_s s_1)/s_1 s_2 = 0$. Clearly $m/s = 0$ iff $m/1 = 0$, so the problem is to show that the kernel of $i_M^S : M \to S^{-1}M$ is equal to $K = \{m \in M : \exists s \in S,\, sm = 0\}$. (c) Show that $S^{-1}M$ is the direct limit of the system formed by the modules $M_f = S_f^{-1}M$ for all $f \in S$, and the canonical homomorphisms $M_f \to M_{fg}$. Deduce that $i_M^S(m) = 0$ iff $i_M^{S_f}(m)$ for some $f \in S$. (d) Prove the isomorphism in (0:1.6.1), directly from the univeral properties of M_f and $\varinjlim M_n$. Deduce that $i_M^{S_f}(m) = 0$ iff $f^n m = 0$ for some n.

4. Let $A = k[x, y]$, where k is an algebraically closed field. (a) Show that if two polynomials $p(x, y)$, $q(x, y)$ have no common factor, then the solution set $V(p, q) \subseteq k^2$ is finite. (b) Deduce that every prime ideal of A is one of the following: (i) the zero ideal, (ii) a maximal ideal $(x - a, y - b)$ for some $(a, b) \in k^2$, or (iii) a principal ideal (f) generated by an irreducible polynomial $f(x, y)$. [For (b)(ii), assume Hilbert's Nullstellensatz: if an ideal $I \subseteq A$ is not the unit ideal, then $V(I) \subseteq k^2$ is non-empty.] Note this shows that Spec(A) consists of closed points in bijection with k^2, together with a generic point for each irreducible algebraic curve in k^2, and a generic point for the whole plane k^2.

5. With A as in Problem 3, let $f(x, y) \in A$ be an irreducible polynomial and $B = k[x, y]/(f)$. (a) Use Problem 3 to show that the points of Spec(B) are (i) the generic point $\mathfrak{p} = (0)$, and (ii) closed points corresponding to points (a, b) on the curve $f(x, y) = 0$ in k^2. (b) Use Problem 3 to show that the closed sets in the the Zariski topology on $X = \mathrm{Spec}(B)$ are X itself, and finite subsets of the closed points. (In particular, any two such curves are homeomorphic in the Zariski topology, but in general not isomorphic as schemes.)

5. Consider the case of Problem 4 where $f(x, y)$ has the form $y - g(x)$, so the curve is the graph of a function $g(x)$. Show that in this case, Spec(B) is isomorphic to the affine line Spec $k[x]$.

6. Let k be a commutative ring and let $A = k[x_{11}, x_{12}, \ldots, x_{NN}, d^{-1}] \underset{\text{def}}{=} S_d^{-1} k[x_{11}, x_{12}, \ldots, x_{NN}]$, where $d(x_{11}, \ldots, x_{NN})$ is the determinant of the $N \times N$ matrix with entries x_{ij}. The affine scheme $GL_N/k = \operatorname{Spec}(A)$ is called the *general linear group scheme* over k. (a) Show that the set of k-algebra homomorphisms $A \to k$ (i.e., morphisms $\operatorname{Spec}(k) \to \operatorname{Spec}(A)$ of schemes over k) corresponds bijectively to the set $GL_N(k)$ of invertible $N \times N$ matrices X over k. (b) Define a morphism $i\colon \operatorname{Spec}(A) \to \operatorname{Spec}(A)$ such that under the correspondence in (a), composition with i corresponds to the map sending a matrix to its inverse. (c) Let $B = k[x_{11}, x_{12}, \ldots, x_{NN}, d(x)^{-1}, y_{11}, y_{12}, \ldots, y_{NN}, d(y)^{-1}]$. Show that k-morphisms $\operatorname{Spec}(k) \to \operatorname{Spec}(B)$ correspond to pairs of invertible matrices (X, Y), so it is natural to define $(GL_N \times GL_N)/k = \operatorname{Spec}(B)$. (When we discuss the general construction of a product of schemes we will see that this is exactly right.) Define a morphism $\mu\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ that corresponds to multiplication in the group $GL_N(k)$.

7. Let k be an algebraically closed field, $B = k[x, y]/(f)$ as in Problem 5, $C = \operatorname{Spec}(B)$ (a *plane curve*), and $T = \operatorname{Spec} k[\delta]/(\delta^2)$. (a) Show that T has only one point. (b) Show that to give a k-morphism $j : T \to C$, it is equivalent to give the following data: (i) the closed point $(a, b) \in k^2$ on C which is the image of j, plus (ii) a tangent vector to C at (a, b), that is, a pair (v_1, v_2) satisfying $v_1 f_x(a, b) + v_2 f_y(a, b) = 0$, where f_x, f_y are the partial derivatives of f (defined formally by the rules for differentiating a polynomial, in case k is not $\mathbb{R}$ or $\mathbb{C}$). (c) Show that $(0, 0)$ is a *singular point* of the curve C defined by $f(x, y) = y^2 - x^3$, in the sense that the space of tangent vectors to C there is 2-dimensional. Deduce that C is not isomorphic to the affine line $\operatorname{Spec} k[x]$.

8. Show that the following are equivalent for an affine scheme $X = \operatorname{Spec}(A)$: (i) X is the union of two disjoint closed (and therefore also open) subsets $V(I_1)$, $V(I_2)$; (ii) there are non-zero idempotents $e_1, e_2 \in A$, such that $e_1 e_2 = 0$ and $e_1 + e_2 = 1$; (iii) the ring A is a cartesian product $A = A_1 \times A_2$. Moreover, the correspondence is such that $\sqrt{I_1} = \sqrt{(e_1)}$, $\sqrt{I_1} = \sqrt{(e_1)}$, $V(I_1) = D(e_2)$ and $V(I_2) = D(e_1)$.

9. Show that $\operatorname{Spec}(A)$ has only one point iff the nilradical of A is maximal, *i.e.*, A is not the zero ring, and every element of A is either invertible or nilpotent.