Homework problems, Set 2

1. Show that the sheaf axiom (F) in (3.1.2) is equivalent to the requirement for every object $T \in K$, the presheaf of sets $\mathcal{G}(U) = \mathrm{Hom}_K(T, \mathcal{F}(U))$ is a sheaf in the usual sense, as remarked in the notes on (3.2.2).

2. The direct image functor f_* for the trivial map $f\colon X \to \{\text{point}\}$ is naturally isomorphic to the global section functor Γ. Use this to give an example in which the induced map on stalks u_x in (3.4.4) is neither injective nor surjective.

*Technically, the first sentence above is only true on the category of presheaves satisfying the condition $\mathcal{F}(\emptyset) = \{0\}$ (for sheaves, this condition holds automatically, by the sheaf axiom). And "naturally isomorphic" should be taken with a grain of salt, since strictly speaking, Γ and f_* are functors to two different, but equivalent, categories.

3. An *adjoint pair of functors* (f, g) consists of (covariant) functors $f\colon K \to L$ and $g\colon L \to K$ between two categories K, L, and a natural isomorphism

$$\mathrm{Hom}_K(-, g(-)) \cong \mathrm{Hom}_L(f(-), -)$$

of functors $K^{\mathrm{op}} \times L \to \underline{\mathrm{Sets}}$. (a) For objects $M \in K$ and $N \in L$, let $\sigma_M\colon M \to g(f(M))$ and $\rho_N\colon f(g(N)) \to N$ correspond to the identity maps $1_{f(M)}$ and $1_{g(N)}$, respectively. Verify that the correspondence is characterized by

$$\mathrm{Hom}_K(M, g(N)) \ni u \leftrightarrow v \in \mathrm{Hom}_L(f(M), N)$$

if and only if $v = \rho_N \circ f(u)$, if and only if $u = g(v) \circ \sigma_M$. The set-up in (3.5.4) is an instance of this. One says that f is *left adjoint* to g, and g is *right adjoint* to f. (b) Show that either functor in an adjoint pair determines the other up to a canonical natural isomorphism. (c) Let g be the "forgetful" functor from groups to sets which assigns to a group G its underlying set. Let f be the functor which assigns to any set X the free group with generators X. Show that (f, g) is an adjoint pair. The same holds with "groups" replaced by "abelian groups" or "commutative rings" (the free commutative ring generated by X is the polynomial ring $\mathbb{Z}[X]$). In fact this holds with groups replaced by any category of "universal algebras," defined as sets equipped with algebraic operations, axiomatized by identities, such as the associative law for multiplication in a group.

4. Deduce the statement in (3.5.5) from the equality $(g \circ f)_* = g_* \circ f_*$ and the adjointness of the inverse and direct image functors.

5. Prove the theorem in (3.6.2).

6. Let X be a topological space with an action of a group G by continuous automorphisms. The group G *acts equivariantly* on a sheaf $\mathcal{F}$ on X if we are given sheaf homomorphisms $\sigma_g\colon \mathcal{F} \to (g^{-1})_*\mathcal{F}$ for all $g \in G$ [so σ_g maps $\mathcal{F}(U)$ to $\mathcal{F}(g(U))$], such that σ_1 is the identity and $\sigma_{gh} = (h^{-1})_*(\sigma_g) \circ \sigma_h$ for all $g, h \in G$. If G also acts on Y, and $f\colon X \to Y$ commutes with the G action, show that $f_*\mathcal{F}$ has a natural equivariant G action.

7. Here is an application of Problem 6. Let $X = Y = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, with the usual Hausdorff topology, and let $G = \{1, \tau\}$ be the two-element group acting on X by $\tau(z) = -z$. Let G act trivially on Y. Let $f\colon X = \mathbb{C}^* \to \mathbb{C}^* = Y$ be given by $f(z) = z^2$. Let $\mathbb{Z}$ denote the constant sheaf on X such that $\mathbb{Z}(U) = \mathbb{Z}$ for every connected U.

(a) Show that there is a unique equivariant action of G on $\mathbb{Z}$ such that under the identification of $\mathbb{Z}(X)$ with $\mathbb{Z}$, σ_τ acts on $\mathbb{Z}(X)$ as multiplication by -1.

(b) Since G acts trivially on Y, Problem 6 gives an action of G by sheaf automorphisms on $f_*\mathbb{Z}$. Let $\mathcal{F} \subseteq f_*\mathbb{Z}$ be the subsheaf of G-invariant sections of $f_*\mathbb{Z}$. Show that $\mathcal{F}$ is a locally constant sheaf with stalks $\mathbb{Z}$, but not a constant sheaf; in fact $\mathcal{F}(Y) = \{0\}$.

(c) Fix a point $z \in Y$, and let $\mathcal{A}$ be the skyscraper sheaf with stalk $\mathbb{Z}$ at z. Verify that there is a surjective sheaf homomorphism $\mathcal{F} \to \mathcal{A}$ that sends s to its germ s_z, or to zero if $s \in \mathcal{F}(U)$ for $z \notin U$. This is an example of a surjective sheaf homorphism which is not surjective on sections, since $\mathcal{A}(Y) = \mathbb{Z}$.

(d) Let $\mathcal{F}$ be a locally constant sheaf on any space X, let z be a point of X, and let $p\colon [0,1] \to X$, $p(0) = p(1) = z$ represent an element of the fundamental group $\pi_1(X, z)$. Then $p^{-1}\mathcal{F}$ is a *constant* sheaf, so there are canonical identifications $\mathcal{F}_z \cong (p^{-1}\mathcal{F})_0 \cong (p^{-1}\mathcal{F})([0,1]) \cong (p^{-1}\mathcal{F})_0 \cong \mathcal{F}_z$. Composing these gives an an automorphism σ_p of $\mathcal{F}_z$, and this is an action of $\pi_1(X, z)$ on $\mathcal{F}_z$, called the *monodromy action*. Show that in the example above, the locally constant sheaf $\mathcal{F}$ on $Y = \mathbb{C}^*$ has non-trivial monodromy; in particular a generator of the fundamental group acts as -1 on $\mathcal{F}_z$.

(e) If you used the Zariski topology on $\mathbb{C}^*$ instead, you would get $\mathcal{F} = 0$. Indeed, since $\mathbb{C}^*$ is irreducible, any locally constant sheaf is constant in the Zariski topology. So, even though we constructed the covering $f\colon X \to Y$ purely algebraically, there is no locally constant sheaf in the Zariski topology on Y that detects its monodromy.

It is possible to define a purely algebraic notion of "sheaf in the étale topology," which for complex algebraic varieties more closely resembles the notion of sheaf in the classical topology, but also makes sense, for example, in characteristic p. Time permitting, we might discuss this later in the course.