Extra problems for winter break

1. Let $X = \mathrm{Proj}(S)$, where $S = k[x, y, z]$ with $\deg(x) = \deg(y) = 1$, $\deg(z) = 2$. Show that the sheaf of modules associated to $S(1)$ is *not* locally free.

2. Given a polynomial ring $S = k[x_0, \ldots, x_n]$, graded with degrees $\deg(x_i) = d_i > 0$, the scheme $X = \mathrm{Proj}(S)$ is called a *weighted projective space* of over k. In particular, ordinary projective space $\mathbb{P}^n_k$ is the special case when all $d_i = 1$.

(a) When k is a field, show that X is a reduced algebraic scheme over k, and when $k = \overline{k}$ is algebraically closed, its closed points can be identified naturally with the orbits of the k^* action on $k^{n+1} \setminus \{\mathbf{0}\}$ given by $t \cdot (z_0, \ldots, z_n) = (t^{d_0} z_0, \ldots, t^{d_n} z_n)$.

(b) Construct a surjective k-morphism $\mathbb{P}^n_k \to X$, such that for any k it is a base change of the case $k = \mathbb{Z}$, and when $k = \overline{k}$ is an algebraically closed field, it is given on closed points by $(z_0 : \cdots : z_n) \mapsto (z_0^{d_0} : \cdots : z_n^{d_n})$, where we also use the colon notation for the orbits identified with closed points of X in part (a). In particular, this identifies X (or at least its set of closed points) with the quotient of $\mathbb{P}^n_k$ by the action of the product of cyclic groups $G = Z_{d_0} \times \cdots \times Z_{d_n}$, acting in the i-th coordinate as multiplication by d_i-th roots of unity.

(c) Prove that when $n = 1$, $X \cong \mathbb{P}^1_k$. [Hint: use the fact that $\mathrm{Proj}(S^{(d)})$ is isomorphic to $\mathrm{Proj}(S)$.]

(d) Prove that when $n = 1$, the morphism in (b) composed with the isomorphism in (c) becomes the morphism $\mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $(z_0 : z_1) \to (z_0^m : z_1^m)$, for some m.

3. Prove that the space of global sections $\Gamma(\mathbb{P}^n_k, \mathcal{O}(m))$ is isomorphic to the degree m homogeneous component S_m of the polynomial ring $S = k[x_0, \ldots, x_n]$ for $m \geq 0$, and is 0 for $m < 0$.

4. Show that $\mathbf{V}(\mathcal{O}(1))$ over $\mathbb{P}^n_k$ is the tautological line bundle, given geometrically as the subvariety of $\mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$ defined by the equations $x_i z_j - x_j z_i = 0$ in homogeneous coordinates z on $\mathbb{P}^n_k$ and affine coordinates x on $\mathbb{A}^n_k$.

5. Let S be a positively graded ring, $A = S_0$, so $X = \mathrm{Proj}(S)$ is a scheme over $Y = \mathrm{Spec}(A)$. Let I be the set of elements $a \in A$ such that S_+ is contained in the radical of the annihilator of a. Prove that I is an ideal, and that $V(I) \subseteq Y$ is the scheme-theoretic closed image of the structure morphism $X \to Y$.