Homework problems, Set 1

0. This is a running exercise for the whole year: make precise all definitions and statements sketched in the EGA outline notes, and fill in omitted proofs. Sometimes I'll give important special cases of "Problem 0" separately, as in some of the problems below.

First some warm-up algebra problems related to EGA 0:1.

1. *Correction*: the problem was supposed to be to prove the "equivalently,..." part of (1.0.5), but I formulated it incorrectly. Here is the right version.

Given an A-algebra B, prove that $b \in B$ is a root of a monic polynomial over A if b is contained in an A-subalgebra $B' \subseteq B$ such that B' is a finitely generated A-module (this is a bit tricky if you don't assume A Noetherian).

2. Show that A is an integral domain iff A is isomorphic to a subring of a field; and A is reduced iff A is isomorphic to a subring of a direct product of fields.

3. Elaborate on (1.3.1): for a multiplicative subset $S \subseteq A$, define how the functor $S^{-1}(-)$ operates on A-module homomorphisms $M \to N$, verify that it is a functor, and prove that it is exact.

4. Prove the last statement in (1.3.5) by using the exactness of $S^{-1}(-)$ and the left-exactness of $\operatorname{Hom}(-, N)$, applied to a presentation $A^p \to A^q \to M \to 0$. Find a counterexample when M is not finitely presented.

5. Liu, exercise 1.2.1.

6. Prove the statement in (1.7.4). First do the case when M is generated by a single element, then apply (1.7.3). Find a counterexample when M is not finitely generated.

7. Prove the statement in (1.7.5). Hint: use Nakayama's lemma [Liu 1.2.7] to show that $M_{\mathfrak{p}}, N_{\mathfrak{p}} \neq 0$ implies $(M \otimes_A N)_{\mathfrak{p}} \neq 0$. Find a counterexample when the finite-generation hypothesis is omitted.

8. Show that if M is finitely generated, and $\mathfrak{a} \subseteq A$ is an ideal, then $\operatorname{Supp}(M/\mathfrak{a}M)$ is the set of primes containing $\mathfrak{a} + \operatorname{ann}(M)$.

Now some geometry problems that you can do in the naive context of affine algebraic sets in k^n (k an algebraically closed field).

9. (a) Show that the parametrization $\phi\colon \langle t \rangle \mapsto \langle t^2, t^3 \rangle$ from the line k^1 to the curve $C = V(y^2 - x^3)$ in k^2, corresponding to the k-algebra homomorphism $\phi^\sharp\colon k[x,y]/(y^2 - x^3) \to k[t]$ defined by $\phi^\sharp(x) = t^2$, $\phi^\sharp(y) = t^3$, is a bijective algebraic map, and a homeomorphism in the Zariski topology. (b) Show that the set of monomials $\{1, y\} \cdot \{1, x, x^2, \ldots\}$ spans $k[x,y]/(y^2 - x^3)$ as a k-vector space, and deduce that $\phi^\sharp$ is injective, $(y^2 - x^3)$ is a radical ideal, and $k[x,y]/(y^2 - x^3)$ is isomorphic to the subring $k[t^2, t^3] \subseteq k[t]$. (c) Conclude that the inverse map to ϕ is not algebraic. This example shows that a bijective algebraic map is not necessarily an isomorphism. (d) Show that the subalgebra $k[t^2, t^3]$ of $k[t]$ is not isomorphic to a polynomial algebra $k[u]$ so k^1 and C are not isomorphic as affine algebraic sets.

10. Consider the plane curve $V(y - f(x)) \subseteq k^2$, *i.e.*, the graph of the polynomial $f(x)$, with coordinate ring $k[x,y]/(y - f(x))$. Show that every k-algebra homomorphism $\phi\colon k[x,y]/(y - f(x)) \to k[\delta]/(\delta^2)$ is given by $\phi(x) = t + a\delta$, $\phi(y) = f(t) + af'(t)\delta$ for unique $t, a \in k$. In other words, the solutions of the equation $y = f(x)$ in the "ring of dual numbers" $k[\delta]/(\delta^2)$ have the form $\langle t, f(t)\rangle + \delta\langle v_1, v_2\rangle$, where $\langle v_1, v_2\rangle$ is a tangent vector to the curve $y = f(x)$ at $\langle t, f(t)\rangle$. (Note that this works even if k is not $\mathbb{R}$ or $\mathbb{C}$, if we define $f'(x)$ according to the usual rule for differentiating a polynomial.) This example shows how solutions of polynomial equations with coordinates in a non-reduced ring can be geometrically meaningful.

11. Prove that a regular function defined on all of the projective space $\mathbb{P}^n(k)$ must be constant. For this you may assume the theorem, which we will prove later, that every regular function on an affine variety $V(I) \subseteq k^n$ is given by a polynomial in the coordinates. (Recall that by definition, f is regular if it is given locally by rational functions in the coordinates, so the theorem just quoted is not obvious.)

12. (a) A *plane conic* is a curve $V(q) \subseteq \mathbb{P}^2(\mathbb{C})$, where $q(x,y,z)$ is a homogeneous quadratic polynomial. Show that for every plane conic C, there is an automorphism of $\mathbb{P}^2(\mathbb{C})$ given by a linear transformation on the homogeneous coordinates x, y, z, which carries C onto one of the following three examples: (i) the "x-axis" $V(y^2) = V(y) \cong \mathbb{P}^1(\mathbb{C})$, (ii) the union of the x and y axes $V(xy)$, or (iii) the projectively completed parabola $V(yz - x^2)$. In particular, the usual classification of non-degenerate conics in the real affine plane $\mathbb{R}^2$ as ellipes, parabolas, or hyperbolas collapses to one single case when we consider their projective completions and allow complex solutions. (b) Show that the classification of non-degenerate conics in the complex affine plane $\mathbb{C}^2$ reduces to two cases: the parabola $V(y - x^2)$, or the hyperbola $V(xy - 1)$, depending on whether the projective completion of C is tangent to the "line at infinity" $V(z)$, or meets $V(z)$ at two distinct points (in particular, there is no distinction between an "ellipse" and a "hyperbola" in $\mathbb{C}^2$).

13. Let k be a field of characteristic not equal to 2. Show that every line of the form $V(y - t(x+1))$ in k^2, where $t \in k$ and $t^2 \neq -1$, meets the conic $V(x^2 + y^2 - 1)$ in exactly two points, $(-1, 0)$ and another point $(a(t), b(t))$. Solve for $a(t)$ and $b(t)$, and show that this gives an algebraic parametrization of the solutions of $x^2 + y^2 = 1$ in k^2 (with $(-1, 0)$ omitted) by the open subset $k^1 \setminus \{\pm\sqrt{-1}\}$ of k^1, even if k is not assumed to be algebraically closed. Deduce that every "Pythagorean triple" of positive integers $a^2 + b^2 = c^2$ has the form $\{2pq, (p+q)(p-q), p^2 + q^2\}$ for some integers $p > q > 0$.