Synopsis of material from EGA I* (1971)
§1: Affine schemes, 1.6*

1.6* Affine schemes and morphisms from locally ringed spaces to affine schemes.

(1.6.1*) *Definition:* An *affine scheme* is a (necessarily locally ringed) ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\mathrm{Spec}(A), \widetilde{A})$ for a ring A. Recall that $\Gamma(X, \mathcal{O}_X) \cong A$ in this case.

(1.6.2*) $\mathrm{Hom}_{\mathrm{loc}}(X, Y)$ denotes the set of *local* morphisms between locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ [Liu, 2.2.20—note that Liu *only* allows local morphisms].

(1.6.3*) *Proposition:* Let $(S, \mathcal{O}_S)$ be an affine scheme and $(X, \mathcal{O}_X)$ any locally ringed space. Then the canonical map

$$\rho\colon \mathrm{Hom}_{\mathrm{loc}}(X, S) \to \mathrm{Hom}_{\mathrm{ring}}(\mathcal{O}_S(S), \mathcal{O}_X(X))$$

(given by the fact that $X \mapsto \mathcal{O}_X(X)$ is a contravariant functor) is *bijective.*

[Let **Loc** be the category of locally ringed spaces and local morphisms, and let **Ring** be the category of commutative rings. Another formulation of Proposition (1.6.3*) is that the functor

$$\mathrm{Spec}\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$$

is right adjoint to the global functions functor

$$X \mapsto \mathcal{O}_X(X)\colon \mathbf{Loc} \to \mathbf{Ring}^{\mathrm{op}}\,.$$

Note that the adjointness property completely determines $\mathrm{Spec}(A)$, that is, the definition of the underlying set, the topology, and the sheaf of rings, as well as the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ induced by a ring homomorphism $B \to A$, are all forced by the requirement that (1.6.3*) should hold.]

(1.6.4*) *Corollary:* A given locally ringed space $(S, \mathcal{O}_S)$ is an affine scheme iff it satisfies the conclusion of (1.6.3*) for all locally ringed spaces $(X, \mathcal{O}_X)$.

(1.6.5*) *Corollary:* The functor Spec is an equivalence of categories from $\mathbf{Ring}^{\mathrm{op}}$ to the category of affine schemes (considered as a full subcategory of the category of locally ringed spaces). The inverse functor is $X \mapsto \mathcal{O}_X(X)$.

[In particular (EGA I, 1.7.2), a morphism of ringed spaces $\phi\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is induced by a ring homomorphism $B \to A$ iff ϕ is local.]

(1.6.6*) [repeats (I:1.6.2)]