SYNOPSIS OF MATERIAL FROM EGA IV
FLATNESS, REGULARITY, AND SMOOTH MORPHISMS
(EXCERPTS FROM 0_{IV}, §17; IV, §2, §6)

0_{IV}, §17: REGULAR RINGS

17.1. **Definition of regular ring.**

Proposition (17.1.1). — *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring,* $\dim(A) = n$, $k = A/\mathfrak{m}$. *The following are equivalent:*

(a) The canonical surjective graded k-algebra homomorphism

$$\phi\colon \mathbf{S}_k(\mathfrak{m}/\mathfrak{m}^2) \to \operatorname{gr}_{\mathfrak{m}}(A) \tag{17.1.1.1}$$

is bijective.

(b) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$.

(c) $\mathfrak{m}$ *can be generated by n elements*

(d) $\mathfrak{m}$ *can be generated by an A-regular sequence*

[(a) $\Leftrightarrow$ (b) holds because $\dim \operatorname{gr}_{\mathfrak{m}}(A) = \dim(A) = n$. The surjectivity of ϕ implies that we always have $\geq$ in (b), hence (c) $\Rightarrow$ (b) by Nakayama, and the converse is clear. An A-regular sequence generating $\mathfrak{m}$ is necessarily a s.o.p., so (d) implies (a–c). Finally, (a) implies that A is an integral domain, so any $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ is regular, and then A/a satisfies (b), so (a–c) $\Rightarrow$ (d) follows by induction on n.]

Definition (17.1.2). — A Noetherian local ring satisfying the conditions in (17.1.1) is *regular.*

Corollary (17.1.3). — *A regular local ring is an integrally closed, Cohen-Macaulay domain.*

[The definition of *Cohen-Macaulay* is the existence of a regular sequence of length n, so this is immediate. Condition (a) in fact implies that A is a unique factorization domain, hence integrally closed.]

Examples (17.1.4). — (i) If $\dim(A) = 0$, then A is regular iff A is a field.

(ii) If $\dim(A) = 1$, then A is regular iff A is a *discrete valuation ring*: the maximal ideal is principal, which implies that all the ideals are powers $\mathfrak{m}^k$. In particular, A is a principal ideal domain.

(iii) The formal power series ring $k[[t_1, \ldots, t_n]]$ over a field k is a regular local ring of dimension n.

Proposition (17.1.5). — *A is regular iff its $\mathfrak{m}$-adic completion $\widehat{A}$ is regular.*

In fact, $\operatorname{gr}_{\mathfrak{m}}(A) \cong \operatorname{gr}_{\mathfrak{m}}(\widehat{A})$.

Definition (17.1.6). — A *regular system of parameters* (r.s.o.p.) in A is a system of parameters that generates the maximal ideal $\mathfrak{m}$.

The existence of an r.s.o.p. is equivalent to A being regular. In that case, any minimal generating set for $\mathfrak{m}$ is an r.s.o.p., but not every s.o.p. need be an r.s.o.p..

Proposition (17.1.7). — Let $(A, \mathfrak{m})$ be a Noetherian local ring, $k = A/m$, $x_1, \dots, x_r$ elements of $\mathfrak{m}$, $\mathfrak{I}$ the ideal they generate. The following are equivalent:

(a) A is regular and the x_i are part of an r.s.o.p.

(a') A is regular and the images $\overline{x_i}$ of x_i in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k.

(b) The x_i are part of an s.o.p. for A, and $A/\mathfrak{I}$ is regular.

Moreover, when these conditions hold, $\mathfrak{I}$ is a prime ideal.

Corollary (17.1.8). — *Let $t \in \mathfrak{m}$. Then (a) A/tA is regular and t is a non-zero-divisor in A, if and only if (b) A is regular and $t \notin \mathfrak{m}^2$.*

Corollary (17.1.9). — *Let $(A, \mathfrak{m})$ be regular, $\mathfrak{I} \subseteq \mathfrak{m}$ an ideal. Then $A/\mathfrak{I}$ is regular iff $\mathfrak{I}$ is generated by part of an r.s.o.p. for A.*

17.2. **Review of injective and projective dimension of modules.**

(17.2.1–9) The *projective dimension* of an A-module M is the minimum length (perhaps infinite) of a (left) projective resolution of M. The *injective dimension* of M is the minimum length of a (right) injective resolution.

[The (classical) right derived functors of $\mathrm{Hom}_A(-,-)$ are denoted $\mathrm{Ext}^i_A(-,-)$. The left derived functors of $(- \otimes_A -)$ are denoted $\mathrm{Tor}^A_i(-,-)$.]

We have $\mathrm{injdim}(M) \leq n$ iff $\mathrm{Ext}^{n+1}(N, M) = 0$ for every N. It suffices that this hold for cyclic modules $N \cong A/\mathfrak{I}$. We have $\mathrm{projdim}(M) \leq= n$ iff $\mathrm{Ext}^{n+1}(M, N) = 0$ for every N. If A is *Noetherian*, it suffices that this hold for cyclic modules N.

The supremum of the projective dimensions of all A-modules is equal to the supremeum of their injective dimensions, and is the smallest n such that $\mathrm{Ext}^{n+1}(M, N) = 0$ for all M, N (or ∞ if there is no such n). It is called the *cohomological dimension* $\mathrm{cohdim}(A)$. If A is Noetherian, it suffices to consider finitely generated modules M, N.

Corollary (17.2.10). — *If A is Noetherian, then*

$$\mathrm{cohdim}(A) = \sup_{\mathfrak{m}}(\mathrm{cohdim}(A_{\mathfrak{p}})) \tag{17.2.10.1}$$

as $\mathfrak{m}$ ranges over $\mathrm{Spec}(A)$ *(and also as $\mathfrak{m}$ ranges over the maximal ideals of A).*

Proposition (17.2.11). — *If A is a Noetherian local ring, k its residue field, then* $\mathrm{cohdim}(A) \leq n$ *iff* $\mathrm{Tor}^A_i(k, k) = 0$ *for $i > n$. It suffices that this hold for $i = n + 1$.*

This is shown by using minimal free resolutions of finitely-generated modules to determine $\mathrm{cohdim}(A)$.

Corollary (17.2.12). — *Let A be a Noetherian ring. Then* $\mathrm{cohdim}(A) \leq n$ *iff* $\mathrm{Tor}^A_i(A/\mathfrak{m}, A/\mathfrak{m}) = 0$ *for $i > n$, for every maximal ideal $\mathfrak{m}$ of A. It suffices that this hold for $i = n + 1$.*

Proposition (17.2.13). — *Let A, B be Noetherian local rings, $\phi\colon A \to B$ a local homomorphism such that B is flat over A. Then*

$$\mathrm{cohdim}(A) \leq \mathrm{cohdim}(B). \tag{17.2.13.1}$$

17.3. **Cohomological theory of regular rings.**

Theorem (17.3.1). — *Let A be a Noetherian local ring. Then A has finite cohomological dimension iff A is regular, in which case*

$$\mathrm{cohdim}(A) = \dim(A). \tag{17.3.1.1}$$

"If" and (17.3.1.1) follow from the Koszul resolution of $k = A/\mathfrak{m}$ for a regular local ring A. For "only if," we proceed by induction on $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$, the case $\mathfrak{m} = 0$ being trivial.

If $\mathfrak{m} \neq 0$ is the annihilator of an element $c \in A$, then $\mathfrak{m}$ is not projective, which implies $\mathrm{cohdim}(A) > 0$. But there is an exact sequence $0 \to k \to A \to E \to 0$, which implies that if $\mathrm{Tor}_n(k, k) \neq 0$ for $n > 0$, then $\mathrm{Tor}_{n+1}(k, E) \neq 0$. Hence $\mathrm{cohdim}(A)$ cannot be finite.

Otherwise, one proves that there is a regular element $a \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then $A' = A/aA$ has finite projective dimension, hence is regular by induction, which implies that A is regular.

"If" is Hilbert's syzygy theorem. "Only if" is a theorem of Serre.

Corollary (17.3.2). — *If A is a regular local ring, then so is $A_{\mathfrak{p}}$, for every $\mathfrak{p} \in \mathrm{Spec}(A)$.*

This follows from (17.2.10) and (17.3.1).

Proposition (17.3.3). — *Let $\phi\colon A \to B$ be a local homomorphism of Noetherian local rings, $\mathfrak{m}$, $\mathfrak{n}$ the maximal ideals of A, B, $k = A/\mathfrak{m}$, $k' = B/\mathfrak{n}$; so ϕ induces a k'-linear homomorphism*

$$\psi\colon (\mathfrak{m}/\mathfrak{m}^2) \otimes_k k' \to \mathfrak{n}/\mathfrak{n}^2. \tag{17.3.3.1}$$

(i) If B is regular and flat over A, then A is regular.

(ii) The following are equivalent:

(a) B is regular and ψ is injective.

(b) A and B are regular and there is an r.s.o.p. (x_i) for A such that $(\phi(x_i))$ is part of an r.s.o.p. for B (in which case, this holds for every *r.s.o.p. in A).*

(c) B and $B \otimes_A k$ are regular, and B is flat over A.

(d) A and $B \otimes_A k$ are regular, and B is flat over A.

(e) A and $B \otimes_A k$ are regular, and $\dim(B) = \dim(A) + \dim(B \otimes_A k)$.

(i) follows from (17.2.13) and (17.3.1).

For (ii), we'll only do "(d) $\Rightarrow$ B is regular". If (x_i) is an r.s.o.p. in A, then it is also B-regular sequence, by flatness. Now (x_i) generates $\mathfrak{m}$, and $B/\mathfrak{m}B$ is regular by hypothesis, hence B is regular by (17.1.7).

[(17.3.4–5) omitted]

(17.5.4). A Noetherian ring A is *regular* if $A_{\mathfrak{p}}$ is a regular local ring for every $\mathfrak{p} \in \mathrm{Spec}(A)$. By (17.3.2), it suffices that this hold for every maximal ideal $\mathfrak{p}$. In particular, this definition agrees with the original one if A is local. It also follows that every localization $S^{-1}A$ is regular if A is regular.

Proposition (17.5.5). — *If A is a regular Noetherian ring, then so is $A[T_1, \ldots, T_n]$.*

The case $B = A[T]$ suffices. Since B is a free A-module, $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat. For $\mathfrak{q} \in \mathrm{Spec}(B)$, $\mathfrak{p} = \mathfrak{q} \cap A$, therefore, $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$. Using (17.3.3), it suffices that $B_q/\mathfrak{p}B_q$

be regular. This ring is a localization of $k[T]$, where $k = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Every local ring $k[T]_{\mathfrak{n}}$, $\mathfrak{n} \in \operatorname{Spec}(k[T])$ is either a field (if $\mathfrak{n} = 0$) or a discrete valuation ring, hence $k[T]$ is regular.

Corollary (17.5.6). — *If A is regular, the formal power series ring $A[[T_1, \dots, T_n]]$ is regular.*

Corollary (17.5.7). — *If A is a quotient of a regular Noetherian ring B, and C is an A-algebra of finite type, then every localization $S^{-1}C$ is a quotient of a regular Noetherian ring.*

§2: BASE-CHANGE AND FLATNESS

2.1. **Flat modules over preschemes.**

(2.1.1). Let $f\colon X \to Y$ be a morphism of preschemes, $\mathcal{F}$ a n $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *f-flat*, or *flat over Y*, at a point $x \in X$ if $\mathcal{F}_x$ is a flat $\mathcal{O}_y$-module, where $y = f(x)$. We say $\mathcal{F}$ is *f-flat*, or *flat over Y* if it is f-flat at every point of X. When $f = 1_X$, we simply write *flat* instead of f-flat.

A morphism $f\colon X \to Y$ is *flat* (at a point x) if $\mathcal{O}_X$ is flat over Y (at x).

Proposition (2.1.2). — *Let $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$, and let $f\colon X \to Y$ correspond to $\phi\colon A \to B$. Let M be a B-module, $\mathcal{F} = \widetilde{M}$. Then $\mathcal{F}$ is flat over Y iff M is a flat A-module.*

Proposition (2.1.3). — *Given $f\colon X \to Y$ and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the following are equivalent:*

(a) For every base extension $Y' \to Y$, the functor $\mathcal{F} \otimes_Y -$ from quasi-coherent $\mathcal{O}_{Y'}$-modules to quasi-coherent $\mathcal{O}_{X_{(Y')}}$-modules is exact.

(a′) Condition (a) holds whenever $Y' \to Y$ is the canonical morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$ *for $y \in Y$.*

(b) $\mathcal{F}$ is flat over Y.

Proposition (2.1.4). — *Given $f\colon X \to Y$, $g\colon Y' \to Y$, and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, let $X' = X_{(Y')} = X \times_Y Y'$, and let $\mathcal{F}' = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}$. Let $g'\colon X' \to X$ be the projection, $x' \in X$ and $x = g'(x')$. If $\mathcal{F}$ is flat over Y at x, then $\mathcal{F}'$ is flat over Y' at x'. In particular, if $\mathcal{F}$ is flat over Y, then $\mathcal{F}'$ is flat over Y', and if f is flat, then $f_{(Y')}$ is flat.*

[(2.1.5), omitted, is essentially the conjunction of the following two corollaries]

Corollary (2.1.6). — *Given $f\colon X \to Y$, $g\colon Y \to Z$, and an $\mathcal{O}_X$-module $\mathcal{F}$, if $\mathcal{F}$ is flat over Y at x, and g is flat at $f(x)$, then $\mathcal{F}$ is flat over Z at x.*

Corollary (2.1.7). — *If $f\colon X' \to X$ and $g\colon Y' \to Y$ are flat S-morphisms, then so is $f \times_S g$.*

Proposition (2.1.8). — *Let $f\colon X \to Y$ be a morphism of preschemes,*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

an exact sequence of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}''$ is flat over Y.

(i) For every $Y' \to Y$ *and quasi-coherent* $\mathcal{O}_{Y'}$*-module* $\mathcal{G}'$*, the functor* $(- \otimes_{\mathcal{O}_Y} \mathcal{G}')$ *applied to the given sequence yields an exact sequence again.*

(ii) $\mathcal{F}$ *is* Y*-flat if and only if* $\mathcal{F}'$ *is.*

Corollary (2.1.9). — *Let* $\mathcal{L}^\bullet$ *be a complex of quasi-coherent* $\mathcal{O}_X$*-modules,* d^i *the* i*-th differential, and suppose that* $\operatorname{im}(d^i)$ *and* $\operatorname{coker}(d^i)$ *are* Y*-flat. Then, with the notation of (2.1.8),* $(- \otimes_{\mathcal{O}_Y} \mathcal{G}')$ *commutes with the* i*-th cohomology functor on* $\mathcal{L}^\bullet$*.*

Corollary (2.1.10). — *If* $\mathcal{L}^\bullet$ *is a flat (left) resolution of a flat* $\mathcal{O}_X$*-module* $\mathcal{F}$*, then applying* $(- \otimes_{\mathcal{O}_Y} \mathcal{G}')$ *yields a flat resolution of* $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}'$*.*

Proposition (2.1.11). — *Let* $f \colon X \to Y$ *be a flat morphism and* $\mathcal{F}$ *a quasi-coherent* $\mathcal{O}_Y$ *module of finite presentation. Let* $\mathcal{I} \subseteq \mathcal{O}_Y$ *be the ideal annihilating* $\mathcal{F}$*. Then* $f^*(\mathcal{I})$ *is the ideal annihilating* $f^*(\mathcal{F})$*.*

Proposition (2.1.12). — *Let* X *be a prescheme,* $\mathcal{F}$ *an* $\mathcal{O}_X$*-module locally of finite presentation at a point* $x \in X$*. Then* $\mathcal{F}_x$ *is a flat* $\mathcal{O}_x$*-module if and only if there is a neighborhood* U *of* x *such that* $\mathcal{F}|U$ *is a locally* $(\mathcal{O}_X|U)$*-module.*

[(2.1.13–14) omitted]

2.3. **Topological properties of flatness.**

(2.3.3). A morphism $f \colon X \to Y$ is *quasi-flat* if there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of *finite type* such that $\operatorname{Supp}(\mathcal{F}) = X$, and $\mathcal{F}$ is flat over Y.

Proposition (2.3.4). — *A quasi-flat morphism* $f \colon X \to Y$ *has the following (equivalent) properties:*

(i) For every $x \in X$ *and every* $y' \in Y$ *such that* $y = f(x) \in \overline{\{y'\}}$*, there exists* $x' \in f^{-1}(y')$ *such that* $x \in \overline{\{x'\}}$*.*

(ii) For every $x \in X$*,* f *maps* $\operatorname{Spec}(\mathcal{O}_{X,x})$ *surjectively on* $\operatorname{Spec}(\mathcal{O}_{Y,y})$*.*

(iii) For every irreducible closed $Y' \subseteq Y$*, every irreducible component of* $f^{-1}(Y')$ *dominates* Y'*.*

Given $\mathcal{F}$ as in (2.3.3) and $x \in X$, $\mathcal{F}_x$ is a non-zero f.g. $\mathcal{O}_x$-module, flat over $\mathcal{O}_y$. Since $\mathcal{O}_y \to \mathcal{O}_x$ is local, Nakayama's lemma implies that $\mathcal{F}_x \otimes_{\mathcal{O}_y} k(y) \neq 0$. Hence $\mathcal{F}_x$ is *faithfully flat* [see below]. In particular, $(\mathcal{F}_x)_{\mathfrak{p}} \neq 0$ for every $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_y)$, which implies (ii), since $\mathcal{F}_x$ is an $\mathcal{O}_x$-module.

[*Lemma* (0_{I}, 6.4.1). — *A flat* A*-module* M *is* faithfully flat*, i.e.,* $M \otimes_A N = 0$ *implies* $N = 0$ *for every* A*-module* N*, if and only if* $M/\mathfrak{m}M \neq 0$ *for every maximal ideal* $\mathfrak{m}$ *of* A*.*]

§6: Flat morphisms of locally Noetherian preschemes

6.1. **Flatness and dimension.**

Proposition (6.1.1). — *Let* $\phi \colon A \to B$ *be a local homomorphism of Noetherian local rings,* $\mathfrak{m}$ *the maximal ideal of* A*,* $k = A/\mathfrak{m}$*. Assume that for every* $\mathfrak{m} \neq \mathfrak{p} \in \operatorname{Spec}(A)$*, no irreducible component of* $\operatorname{Spec}(B/\mathfrak{p}B)$ *lies over* $\mathfrak{m}$*. Then*

$$\dim(B) = \dim(A) + \dim(B \otimes_A k). \tag{6.1.1.1}$$

If $\dim(A) = 0$ this is easy. If $\dim(A) > 0$, we can find $a \in \mathfrak{m}$ not contained in any minimal prime of A, nor (by the hypothesis) in $\phi^{-1}(\mathfrak{q})$ for any minimal prime $\mathfrak{q}$ of B. Then $A' = A/aA$, $B' = B/aB$ again satisfy the hypothesis, and $\dim(A') = \dim(A) - 1$, $\dim(B') = \dim(B) - 1$, $B' \otimes_{A'} k = B \otimes_A k$, so the result follows by induction on $\dim(A)$.

Corollary (6.1.2). — *Let $\phi\colon A \to B$ be a local homomorphism of Noetherian local rings, $k = A/\mathfrak{m}$ the residue field. Let $M \neq 0$ be a f.g. A-module, $N \neq 0$ a f.g. B-module. If N is a flat A-module, then*

$$\dim_B(M \otimes N) = \dim_A(N) + \dim_{B \otimes_A k}(N \otimes_A k). \tag{6.1.2.1}$$

In particular, taking $M = A$, $N = B$, if B is flat over A, then (6.1.1.1) holds.

Modding out their annihilators, we can assume $\mathrm{Supp}(M) = \mathrm{Spec}(A)$, and then w.o.l.o.g. $M = A$, and $\mathrm{Supp}(N) = \mathrm{Spec}(B)$, which means that $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is *quasi-flat* (2.3.3). Then (2.3.4) implies that the hypothesis of (6.1.1) holds.

6.5. **Flatness and property (R_k).**

Proposition (6.5.1). — *Let A, B be Noetherian local rings, k the residue field of A, $\phi\colon A \to B$ a local homomorphism, such that B is flat over A.*

(i) If B is regular, then A is regular.

(ii) If A and $B \otimes_A k$ are regular, then B is regular.

This just restates part of (0, 17.3.3).

Corollary (6.5.2). — *Let $f\colon X \to Y$ be a flat morphism of locally Noetherian preschemes.*

(i) If X is regular at x, then Y is regular at $f(x)$.

(ii) If Y is regular at $y \in f(X)$, and $f^{-1}(y)$ is regular at x, then X is regular at x.

6.6. **Transitivity properties.**

Proposition (6.6.1). — [part (c)] *Let X, Y, Z be loclly Noetherian preschemes, $f\colon X \to Y$, $g\colon Y \to Z$ two morphisms.*

(i) Suppose f and g are flat, and for every $y \in f(X)$ (resp. $z \in g(Y)$), the fiber $f^{-1}(y)$ (resp. $g^{-1}(z)$) is regular. Then $h = g \circ f$ is flat, and for every $z \in h(X)$, the fiber $h^{-1}(z)$ is regular.

(ii) Suppose f is faithfully flat [= flat and surjective], $h = g \circ f$ is flat, and every fiber of f and every fiber of h is regular. Then every fiber of g is regular.

6.7. **Application to base-change for algebraic preschemes.**

Proposition (6.7.4). — [part (a)] *Let k be a field, X a locally Noetherian k-prescheme, $k' \supseteq k$ a field extension, $X' = X \otimes_k k'$, x' a point of X', x the image of x' under the projection $X' \to X$. Suppose either that X is locally of finite type over k, or that k' is a finitely generated extension of k.*

If X' is regular at x', then X is regular at x. If k' is separable over k, the converse holds.

Using (6.5.2), this reduces to the following lemma.

Lemma (6.7.4.1). — Let K, L be extensions of a field k, either one of which is finitely generated. If L is a separable extension of k, then $K \otimes_k L$ is regular.

Remark (6.7.5). — Without the hypothesis that $k' \supseteq k$ is separable, X' may not even be reduced at x'. EGA also discusses an example in which X is a curve, and X' is reduced and integral at x', but X' is not normal, and thus not regular.

Definition (6.7.6). — Let k be a field, X a locally Noetherian k-prescheme. X is *geometrically regular* at x if, for every finite extension $k' \supseteq k$, $X' = X \otimes_k k'$ is regular at each point x' lying over x. X is *geometrically regular* if it is so at every point.

A Noetherian k-algebra A is *geometrically regular* if $\mathrm{Spec}(A)$ is.

If $X = \mathrm{Spec}(K)$ where K is an field extension of k, then X is geometrically regular iff $K \supseteq k$ is a separable extension.

Proposition (6.7.7). — *Let k be a field, X a locally Noetherian k-prescheme, $x \in X$. Let $R(k')$ denote the property of an extension $k' \supseteq k$, that $X' = X \otimes_k k'$ is regular at every point x' lying over x. The following are equivalent:*

(a) $R(k')$ holds for every finite extension k' of k.

*(b) $R(k')$ holds for every finite radicial extension k' of k [*radicial *means* $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ *is universally injective].*

(c) $R(k')$ holds for every finitely generated extension k' of k.

If X is locally of finite type over k, then the above are also equivalent to the following:

(d) $R(k')$ holds for every extension k'.

(e) $R(k')$ holds for some *perfect extension k'.*

(f) $R(k')$ holds for every extension of the form $k' = k^{p^{-n}}$, where $p = \mathrm{char}(k)$.

Corollary (6.7.8). — [part (iv)] *Let k be a field, X a locally Noetherian k-prescheme, $x \in X$. Let k' be an extension of k and assume either that k' is a finitely generated extension, or X is of finite type over k, so $X' = X \otimes_k k'$ is locally Noetherian. If $x' \in X'$ lies over $x \in X$, then X is geometrically regular at x if and only if X' is geometrically regular at x'.*

6.8. **Regular, [normal, reduced,] and smooth morphisms.**

Definition (6.8.1). — Let $f\colon X \to Y$ be a morphism such that $f^{-1}(y)$ is locally Noetherian for every $y \in Y$, and x a point of X. Then f is *regular* at x if f is flat at x, and $f^{-1}(f(x))$ is *geometrically* regular at x. We say f is *smooth* at x if it is regular and locally of finite presentation at x.

Proposition (6.8.2). — *Let $f' = f_{(Y')}$ be a base extension of $f\colon X \to Y$, x' a point of $X' = X_{(Y')}$, x the image of x' in X. Assume that either f or the base extension $Y' \to Y$ is locally of finite type. If f is regular (resp. smooth) then so is f'.*

Proposition (6.8.3). —

(i) Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of locally Noetherian preschemes. If f and g are regular, then so is $g \circ f$.

(ii) If f is surjective, and f and $g \circ f$ are regular, then so is g.

(iii) Let Y' *be another locally Noetherian prescheme,* $h\colon Y' \to Y$ *a morphism, and set* $X' = X \times_Y Y'$, $f' = f_{(Y')}\colon X' \to Y'$. *Assume that* f *or* h *is locally of finite type. If* f *is regular, then so is* f'*; and the converse holds if* h *is faithfully flat.*

Assertions (i) and (iii) also hold with "smooth" in place of "regular," under the additional assumption in (iii) that h *is quasi-compact.*

Remark (6.8.4). — If X is a locally Noetherian k-prescheme, then, by definition, X is regular over k iff $X \to k$ is geometrically regular. This is a stronger condition than for X to be a regular scheme (a property which is independent of k).

Proposition (6.8.5). — *Let* X, Y *be locally Noetherian* k*-preschemes (*k *a field) one of which is locally of finite type over* k.

(i) If X *is regular and* Y *is geometrically regular, then* $X \times_k Y$ *is regular.*

(ii) If X *and* Y *are geometrically regular, then* $X \times_k Y$ *is geometrically regular.*

Theorem (6.8.6). — *Let* $f\colon X \to Y$ *be a morphism locally of finite type between locally Noetherian preschemes,* $x \in X$, $y = f(x)$. *The following are equivalent:*

(a) f *is smooth at* x.

(b) f *is regular at* x.

(c) $\mathcal{O}_x$ *is a formally smooth* $\mathcal{O}_y$ *algebra [this is defined in (0, 19.3.1)] for the pre-adic topologies on* $\mathcal{O}_x$ *and* $\mathcal{O}_y$.

(c′) $\mathcal{O}_x$ *is a formally smooth* $\mathcal{O}_y$ *algebra for the discrete topologies on* $\mathcal{O}_x$ *and* $\mathcal{O}_y$.

Corollary (6.8.7). — *Let* $f\colon X \to Y$ *be a morphism locally of finite type between locally Noetherian preschemes. Then the set of points* $x \in X$ *where* f *is smooth (or regular) is open.*

Remark (6.8.8). — It will be shown in (17.5.1) that (b) $\Leftrightarrow$ (c′) in (6.8.6) and hence also (6.8.7) hold without the Noetherian hypothesis, provided f is locally of finite presentation.