SYNOPSIS OF MATERIAL FROM EGA IV
§5: DIMENSION, DEPTH AND REGULARITY FOR LOCALLY NOETHERIAN PRESCHEMES, 5.1–6

5.1. **Dimension of preschemes.**

(5.1.1). Recall from (0, 14.1) and (0, 16.1) that for any ring A and ideal $\mathcal{I} \subseteq A$,

$$\dim(\operatorname{Spec}(A)) = \dim(A), \tag{5.1.1.1}$$

$$\dim(V(\mathcal{I})) = \dim(A/\mathcal{I}), \tag{5.1.1.2}$$

$$\operatorname{codim}(V(\mathcal{I}), \operatorname{Spec}(A)) = \operatorname{ht}(\mathcal{I}). \tag{5.1.1.3}$$

(5.1.1.4) $\operatorname{Spec}(A)$ is catenary iff A is catenary.

(5.1.1.5) $\operatorname{Spec}(A)$ is equidimensional iff A is equidimensional, iff $\dim(A/\mathfrak{p})$ is constant all for minimal primes $\mathfrak{p}$.

(5.1.1.6) $\operatorname{Spec}(A)$ is equicodimensional iff A is equicodimensional, iff all maximal ideals of A have the same height.

Recall that A is *biequideminsional* if $\operatorname{Spec}(A)$ is Noetherian, and A is equidimensional, equicodimensional, catenary, and of finite dimension.

Proposition (5.1.2). — *Let $Y = \overline{\{y\}} \subseteq X$ be irreducible. Then*

$$\operatorname{codim}(Y, X) = \dim(\mathcal{O}_{X,y}). \tag{5.1.2.1}$$

Corollary (5.1.3). — *For any closed $Y \subseteq X$,*

$$\operatorname{codim}(Y, X) = \inf_{y \in Y} \dim(\mathcal{O}_{X,y}), \tag{5.1.3.1}$$

and if X is locally Noetherian, then for all $x \in Y$,

$$\operatorname{codim}_x(Y, X) = \inf_{y \in Y, x \in \overline{\{y\}}} \dim(\mathcal{O}_{X,y}). \tag{5.1.3.2}$$

We take (5.1.3.1) as the definition of $\operatorname{codim}(Y, X)$ for an arbitrary subset $Y \subseteq X$.

Proposition (5.1.4). — *For any prescheme X, we have*

$$\dim(X) = \sup_{x \in X} \dim(\mathcal{O}_{X,x}). \tag{5.1.4.1}$$

If every irreducible closed subset of X has a closed point, the supremum may be taken over closed points x.

Corollary (5.1.5). — *X is cateneray iff every local ring $\mathcal{O}_{X,x}$ is catenary. If every irreducible closed subset contains a closed point, it suffices that $\mathcal{O}_{X,x}$ be catenary for closed points x.*

(5.1.6). Recall (0, 16.2.3) that a Noetherian local ring has finite dimension, equal to the minimum number of generators of an ideal of definition.

Proposition (5.1.7). — *If X is locally Noetherian and $Y \subseteq X$ is non-empty, then* $\operatorname{codim}(Y, X)$ *is finite. If X is affine and Noetherian, and Y is irreducible, then* $\operatorname{codim}(Y, X)$ *is the minimum number of sections $s_i \in \Gamma(X, \mathcal{O}_X)$ such that Y is a component of $V(s_1, \ldots, s_n)$.*

Corollary (5.1.8). — *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on a locally Noetherian prescheme X, $f \in \Gamma(X, \mathcal{L})$, $Z = \{x \in X : f(x) = 0\}$. Then every component of Z has codimension ≤ 1 in X, with equality if Z does not contain an irreducible component of X.*

Proposition (5.1.9). — *Let X be locally Noetherian, $Y \subseteq X$ closed, $x \in Y$ such that $\mathcal{O}_{X,x}$ is catenary. Then*

$$\operatorname{codim}_x(Y, X) = \dim(\mathcal{O}_{X,x}) - \operatorname{codim}(\overline{\{x\}}, Y) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,x}). \tag{5.1.9.1}$$

Proposition (5.1.10). — *(i) For any prescheme X, every non-empty, locally constructible subset E contains a point x such that x is isolated in $\overline{\{x\}}$, i.e., $\{x\}$ is locally closed.*

(ii) Let X be locally Noetherian, $\{x\}$ locally closed. Then $\dim(\overline{\{x\}}) \leq 1$, i.e., every point $y \neq x$ of $\overline{\{x\}}$ is closed.

[(i) comes down to the fact that every affine scheme has a closed point. (ii) follows from (0, 16.3.3), applied to $A = \mathcal{O}_{Z,z}$, where $Z = \overline{\{x\}}$ with the reduced subscheme structure, and $z \in Z$, since $\{x\}$ open in Z implies $\{x\} = D(f)$ for some $f \in A$, and then A_f is a field.]

Corollary (5.1.11). — *If X is locally Noetherian, then every closed subset of X contains a closed point.*

[The *constructible sets* are the members of the Boolean algebra generated by open sets U such that the inclusion $U \hookrightarrow X$ is quasi-compact. If X is locally Noetherian, all open sets have this property, so all closed subsets are constructible.]

(5.1.12). Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type, so $S = \operatorname{Supp}(\mathcal{F})$ is closed. Identify S with the underlying space of a closed subscheme of X (not necessarily reduced). For any $x \in X$, $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module, and

$$\dim(\mathcal{F}_x) = \dim(\mathcal{O}_{S,x}), \tag{5.1.12.1}$$

or equivalently,

$$\dim(\mathcal{F}_x) = \operatorname{codim}(\overline{\{x\}}, S). \tag{5.1.12.2}$$

We say $\mathcal{F}$ is *equidimensional at x* if $\mathcal{F}_x$ is equidimensional as an $\mathcal{O}_{X,x}$-module, or equivalently, $\mathcal{O}_{S,x}$ is equidimensional.

We define $\dim(\mathcal{F}) = \dim(\operatorname{Supp}(\mathcal{F}))$; then

$$\dim(\mathcal{F}) = \sup_x \dim(\mathcal{F}_x). \tag{5.1.12.3}$$

If $X = \operatorname{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$, where M is a finitely generated A-module, then $\dim(\mathcal{F}) = \dim(M)$.

Proposition (5.1.13). — *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type, $x \in \overline{\{x'\}} \subseteq X$. Then $\dim(\mathcal{F}_{x'}) \leq \dim(\mathcal{F}_x)$.*

5.2. **Dimension of an algebraic prescheme.**

Proposition (5.2.1). — *[Liu, 2.5.19] Let X be irreducible and locally of finite type over a field k, with generic point ξ. Then X is biequidimensional and* $\dim(X) = \operatorname{trdg}_k k(\xi)$.

Corollary (5.2.2). — *For any prescheme X locally of finite type over k,* $\dim(X) = \sup_x \operatorname{trdg}_k k(x)$, *the supremum taken over the generic points of the components of X.*

Corollary (5.2.3). — *For X locally of finite type over k, $x \in X$,*

$$\dim_x(X) = \dim(\mathcal{O}_{X,x}) + \operatorname{trdg}_k k(x). \tag{5.2.3.1}$$

Corollary (5.2.4). — *Let X be locally of finite type over k, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, $f \in \Gamma(X, \mathcal{L})$, and suppose $Y = \{x \in X : f(x) = 0\}$ is nowhere dense [i.e., doesn't contain a component of X]. Then* $\dim(Y) \leq \dim(X) - 1$, *with equality if Y meets every irreducible component of maximal dimension in X.*

Remarks (5.2.5). — (i) Not every locally Noetherian prescheme is catenary, so this is a special property of algebraic k-preschemes, or more generally of preschemes X such that every $\mathcal{O}_{X,x}$ is a quotient of a regular local ring (e.g., if X is regular). Futhermore, even if A is a regular domain, $X = \operatorname{Spec}(A)$ need not be biequidimensional. Example: let $A = B[t]$, where B is a discrete valuation ring. If (f) is the maximal ideal of B, then in A we have $\operatorname{ht}((f,t)) = 2$, $\operatorname{ht}(ft - 1) = 1$, and these are both maximal ideals, since B_f is a field.

(ii) If X is locally of finite type over k, and $U \subseteq X$ is a dense open subset, then $\dim(U) = \dim(X)$. This does not hold in general, even for a regular, biequidimensional, Noetherian prescheme. Example: if B is a DVR, $X = \operatorname{Spec}(B)$, then $U = \{(0)\}$ is open and dense, $\dim(U) = 0$, $\dim(X) = 1$.

5.3. **Dimension of the support of a sheaf of modules and the Hilbert polynomial.**

Proposition (5.3.1). — *Let A be an artinian local ring, X projective over* $\operatorname{Spec}(A)$, *$\mathcal{L}$ an $\mathcal{O}_X$-module very ample relative to* $\operatorname{Spec}(A)$, *$\mathcal{F} \neq 0$ a coherent $\mathcal{O}_X$-module, $\mathcal{F}(n) = \mathcal{F} \otimes_X \mathcal{L}^{\otimes n}$. Then the degree of the Hilbert polynomial $P(n) = \chi_A(\mathcal{F}(n))$ is equal to* $\dim(\operatorname{Supp}(\mathcal{F}))$.

[The definition of the Hilbert polynomial and the proof use sheaf cohomology (EGA III). This result is not used below.]

5.4. **Dimension of the image of a morphism.**

Proposition (5.4.1). — *Let $f \colon X \to Y$ be a morphism of locally Noetherian preschemes.*

(i) If f is quasi-finite, then $\dim(X) \leq \dim(\overline{f(X)}) \leq \dim(Y)$.

(ii) If f is surjective and open, or surjective and closed, then $\dim(X) \geq \dim(Y)$.

[*Quasi-finite* means f is of finite type, and its fibers are discrete, hence finite (II, 6.2). (i) follows from (0, 16.3.10).]

Corollary (5.4.2). — *If $f \colon X \to Y$ is a finite (hence, closed) morphism of locally Noetherian preschemes, then* $\dim(X) = \dim(f(X))$.

[*Finite* means f is affine, and $f_*\mathcal{O}_X$ is of finite type as an $\mathcal{O}_Y$-module. A finite morphism is both quasi-finite and closed.]

Remarks (5.4.3). — (i) If $f\colon X \to Y$ is a k-morphism of schemes locally of finite type over k, and is quasi-compact and dominant, then $\dim(Y) \leq \dim(X)$. But if X and Y are merely locally Noetherian, it is even possible for f to be of finite type, bijective, and a local immersion, yet have $\dim(Y) > \dim(X)$. Example: let A be a DVR with fraction field K and residue field k, $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(K) \bigsqcup \mathrm{Spec}(k)$.

(ii) (5.4.2) generalizes the result (0, 16.1.5) that if A is a subring of B and B is module-finite over A, then $\dim(A) = \dim(B)$.

5.5. **Dimension formula for a morphism of finite type.**

(5.5.1). The following proposition follows from (0, 16.3.9.1).

Proposition (5.5.2). — *[Liu, 4.3.12] Let $f\colon X \to Y$ be a morphism of locally Noetherian preschemes, $x \in X$, $y = f(x)$. Then*

(5.5.2.1) $$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{Y,y}) + \dim(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y)).$$

In particular, if x is a generic point of a component of $f^{-1}(y)$, then

(5.5.2.2) $$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{Y,y}).$$

Proposition (5.5.3). — *Let A be Noetherian, $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathfrak{p}' = \mathfrak{p}A[t] \in \mathrm{Spec}(B)$, where $B = A[t]$ (note that $\mathfrak{p}' \cap A = \mathfrak{p}$). There are infinitely many prime ideals $\mathfrak{q} \subseteq B$, $\mathfrak{q} \neq \mathfrak{p}'$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. None of these ideals $\mathfrak{q}$ contains another, and they satisfy*

(5.5.3.1) $$\dim(B_{\mathfrak{q}}) = \dim(B_{\mathfrak{p}'}) + 1 = \dim(A_{\mathfrak{p}} + 1)$$

[Proof: Using (5.5.1), one can reduce to the case $\mathfrak{p} = 0$. Then if K is the fraction field of A, the desired primes $\mathfrak{q}$ are identified with the closed points of $\mathrm{Spec}(K[t])$.]

Corollary (5.5.4). — *For any Noetherian ring A,*

(5.5.4.1) $$\dim(A[t_1, \ldots, t_r]) = \dim(A) + r.$$

There are examples of non-Noetherian local rings A such that $\dim(A) = 1$, $\dim(A[t]) = 3$.

Corollary (5.5.5). — *If X is a locally Noetherian prescheme, then $\mathbb{A}^r(X) = X \otimes_{\mathbb{Z}} \mathbb{Z}[t_1, \ldots, t_r]$ has dimension $\dim(X) + r$.*

Corollary (5.5.6). — *In (5.5.3), for any $\mathfrak{q} \in \mathrm{Spec}(A[t])$ such that $\mathfrak{q} \cap A = \mathfrak{p}$, if k, k' are the residue fields of $A_{\mathfrak{p}}$, $B_{\mathfrak{q}}$ respectively, then*

(5.5.6.1) $$\dim(A_{\mathfrak{p}}) + 1 = \dim(B_{\mathfrak{q}}) + \mathrm{trdg}_k\, k'.$$

Lemma (5.5.7). — *Let $(A, \mathfrak{m})$ be a Noetherian local domain, $k = A/\mathfrak{m}$.*

(i) For any integral domain $B \supseteq A$ generated as an A-algebra by one element x, and any $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{m}$,

(5.5.7.1) $$\dim(A) + \mathrm{trdg}_A B \geq \dim(B_{\mathfrak{q}}) + \mathrm{trdg}_k\, k',$$

where k' is the residue field of $B_{\mathfrak{q}}$ and by $\mathrm{trdg}_A B$ we mean the transcendence degree of the fraction field of B over that of A.

(ii) Suppose that for every maximal ideal $\mathfrak{n}$ *of* $A[t]$ *such that* $\mathfrak{n} \cap A = \mathfrak{m}$*, the ring* $(A[t])_{\mathfrak{n}}$ *is catenary. Then equality holds in (5.5.7.1).*

Theorem (5.5.8). — ("dimension formula") *Let* $(A, \mathfrak{m})$ *be a local Noetherian domain,* $B \supseteq A$ *an integral domain finitely generated as an* A*-algebra,* $\mathfrak{q} \in \operatorname{Spec}(B)$ *such that* $\mathfrak{q} \cap A = \mathfrak{m}$*,* $k = A/\mathfrak{m}$*,* $k' = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$*. Then*

$$\dim(A) + \operatorname{trdg}_A B \geq \dim(B_{\mathfrak{q}}) + \operatorname{trdg}_k k'. \tag{5.5.8.1}$$

Suppose futher that every finitely generated A*-subalgebra* A' *of* $B[t]$ *has the property that* $A'_{\mathfrak{m}'}$ *is catenary for every maximal ideal* $\mathfrak{m}' \subseteq A'$ *such that* $\mathfrak{m}' \cap A = \mathfrak{m}$*,. Then equality holds in (5.5.8.1).*

5.6. **Dimension formula and universally catenary rings.**

Proposition (5.6.1). — *Let* A *be a Noetherian ring. The following are equivalent:*

(a) Every polynomial ring $A[t_1, \ldots, t_n]$ *is catenary.*

(b) Every finitely generated A*-algebra is catenary.*

(c) A *is catenary, and for every local* A*-algebra* $(B', \mathfrak{q})$ *which is a localization of a finitely generated* A*-algebra and an integral domain, letting* $\mathfrak{p}$ *denote the preimage of* $\mathfrak{q}$ *in* A*, and* A' *the image of* $A_{\mathfrak{p}}$ *in* B'*, there holds*

$$\dim(A') + \operatorname{trdg}_{A'} B' = \dim(B') + \operatorname{trdg}_k k', \tag{5.6.1.1}$$

where k*,* k' *are the residue fields of* A'*,* B'*.*

Definition (5.6.2). — A Noetherian ring A satifying the conditions in (5.6.1) is *universally catenary.*

Remarks (5.6.3). — (i) If A is universally catenary, then so is $S^{-1}A$; conversely if every $A_{\mathfrak{p}}$ is universally catenary, then so is A.

(ii) A prescheme X is called *univerally catenary* if every $\mathcal{O}_{X,x}$ is universally catenary.

(iii) Criterion (c) depends only on the quotients of A which are integral domains. Hence if $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the minimal primes of A, then A is universally catenary iff each $A/\mathfrak{p}_i$ is, and X is universally catenary iff X_{red} is.

(iv) Criterion (b) and remark (i) show that any localization of a finitely generated A-algebra is universally catenary.

Proposition (5.6.4). — *Any quotient of a regular ring is universally catenary.*

More generally, any quotient of a Cohen-Macaulay ring is universally catenary.

Proposition (5.6.5). — *Let* $f\colon X \to Y$ *be a dominant morphism locally of finite type, where* X *and* Y *are irreducible and* Y *is locally Noetherian (hence so is* X*). Let* ξ*,* η *be the generic points of* X*,* Y*, and let* $e = \dim(f^{-1}(\eta)) = \operatorname{trdg}_{k(\eta)} k(\xi)$ ("dimension of the generic fiber")*. For every* $x \in X$*, with* $y = f(x)$*, we have*

$$e + \dim(\mathcal{O}_{Y,y}) \geq \operatorname{trdg}_{k(y)} k(x) + \dim(\mathcal{O}_{X,x}) \tag{5.6.5.1}$$

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{Y,y}) + \dim(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y)) - \delta(x), \tag{5.6.5.2}$$

where $\delta(x) = \dim_x(f^{-1}(y)) - e$*. If* Y *is universally catenary, then equality holds. If in addition,* x *is closed in* $f^{-1}(y)$*, then*

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) + e. \tag{5.6.5.3}$$

[This follows from preceding results in the universally catenary case. The general inequality and also that $\delta(x) \geq 0$ are proved in (IV, 13).]

Corollary (5.6.6). — *Under the general hypotheses of (5.6.5) we have*

$$\dim(X) \leq \dim(Y) + e. \tag{5.6.6.1}$$

If Y *is universally catenary, then equality holds iff*

$$\dim(Y) = \sup_{y \in f(X)} \dim(\mathcal{O}_{Y,y}). \tag{5.6.6.2}$$

In particular, equality holds in (5.6.6.1) if Y *is locally of finite type over a field.*

Corollary (5.6.7). — *Let* Y *be a locally Noetherian prescheme,* $f\colon X \to Y$ *a morphism locally of finite type. Suppose that* $\dim(f^{-1}(y)) \leq n$ *for all* $y \in Y$*. Then*

$$\dim(X) \leq \dim(Y) + n. \tag{5.6.7.1}$$

Corollary (5.6.8). — *Let* Y *be a locally Noetherian prescheme,* $f\colon X \to Y$ *a morphism locally of finite type. Suppose that* X *is irreducible,* Y *is universally catenary, and* $\dim(f^{-1}(y)) \geq n$ *for all* $y \in Y$*. Then*

$$\dim(X) \geq \dim(Y) + n, \tag{5.6.8.1}$$

with equality if $\dim(f^{-1}(y)) = n$ *for all* $y \in Y$*.*

Remarks (5.6.9). — (i) We need not have equality in (5.6.6.1) even if Y is regular, X is irreducible, and f is dominant and of finite type. Example: $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$, where K is the fraction field of a DVR A.

(ii) Example (5.4.3, (i)) shows that the hypothesis that X is irreducible is needed in (5.6.6) and (5.6.8).

Proposition (5.6.10). — *Let* A *be a universally catenary local Noetherian domain,* $B \supseteq A$ *a domain finite over* A*. Then* $\dim(B_{\mathfrak{n}}) = \dim(A)$ *for every maximal ideal* $\mathfrak{n} \subseteq B$*.*

Example (5.6.11). — The conclusion of (5.6.10) also holds if A is integrally closed rather than universally catenary, by (0, 16.1.6). But there exist catenary local Noetherian domains A for which (5.6.10) fails [details omitted].