

SYNOPSIS OF MATERIAL FROM EGA IV

§20: MEROMORPHIC AND PSEUDO-MEROMORPHIC FUNCTIONS, 20.1.11–20.2

§21: DIVISORS, 21.4, 6

20.1. **Meromorphic functions.** (continued)

(20.1.11–13) Let $(f, \theta): X' \rightarrow X$ be a morphism of ringed spaces, $U \subseteq X$ open, $\mathcal{S}_f(U)$ the set of regular elements $s \in \Gamma(U, \mathcal{O}_X)$ such that $\theta^\sharp(s) \in \Gamma(f^{-1}(U), \mathcal{O}_{X'})$ is also regular. Then \mathcal{S}_f is a subsheaf of $\mathcal{S}(\mathcal{O}_X)$. Define $\mathcal{M}_f = \mathcal{O}_X[\mathcal{S}_f^{-1}]$, the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)[\mathcal{S}_f(U)^{-1}]$, a subsheaf of \mathcal{O}_X -algebras in \mathcal{M}_X . There is a canonical homomorphism of $\mathcal{O}_{X'}$ -algebras $f^*(\mathcal{M}_f) \rightarrow \mathcal{M}_{X'}$. Write $\phi \circ f$ for the image of $f^*\phi$, where ϕ is a section of \mathcal{M}_f .

More generally, define $\mathcal{M}_f(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_f$, and write $u \mapsto u \circ f$ for the homomorphism

$$\Gamma(X, \mathcal{M}_f(\mathcal{F})) \rightarrow \Gamma(X', \mathcal{M}_{X'}(f^*\mathcal{F})).$$

We have

$$f^{-1}(\text{dom}(u)) \subseteq \text{dom}(u \circ f),$$

where $\text{dom}(u)$ denotes the domain of definition of u , *i.e.*, the largest open set on which the restriction of u belongs to the image of \mathcal{F} in $\mathcal{M}_f(\mathcal{F})$.

If $\mathcal{F} = \mathcal{L}$ is invertible and $\mathcal{M}_f = \mathcal{M}_X$, then $u \circ f$ is a regular meromorphic section of $f^*\mathcal{L}$ if u is a regular meromorphic section of \mathcal{L} .

Given another morphism $f' : X'' \rightarrow X$, if $\mathcal{M}_f = \mathcal{M}_X$ and $\mathcal{M}_{f'} = \mathcal{M}_{X'}$, then $\mathcal{M}_{f \circ f'} = \mathcal{M}_X$, and for any meromorphic section u of \mathcal{F} we have $u \circ (f \circ f') = (u \circ f) \circ f'$.

Proposition (20.1.14). — *If $f: X' \rightarrow X$ is flat, then $\mathcal{M}_f = \mathcal{M}_X$, so $\phi \circ f$ is defined for all $\phi \in M(X)$. If f is a flat morphism of locally ringed spaces, then $\text{dom}(\phi \circ f) = f^{-1}(\text{dom}(\phi))$, and if in addition f is surjective, then $\phi \mapsto \phi \circ f$ is injective.*

Remark (20.1.15). — Our notion of meromorphic function coincides with the usual one when X is a reduced complex analytic space. If Y is a prescheme locally of finite type over \mathbb{C} , its closed points form a complex analytic space Y^{an} , and the canonical map $f: Y^{\text{an}} \rightarrow Y$ is flat, hence $u \rightarrow u \circ f$ from $M(Y)$ to $M(Y^{\text{an}})$ is defined for all u and injective; but not surjective in general. For this reason it is preferable in algebraic geometry to use the (equivalent, in this setting) terminology of “pseudo-functions” defined in the next section.

20.2. **Pseudo-morphisms and pseudo-functions.**

(20.2.1–3) An open subset U of a prescheme X is *schematically dense* if $\Gamma(V, \mathcal{O}_X) \rightarrow \Gamma(V \cap U, \mathcal{O}_X)$ is injective for all open V (equivalently, $\mathcal{O}_X \rightarrow j_*j^{-1}\mathcal{O}_X$ is injective, where $j: U \hookrightarrow X$ is the inclusion). A *strict rational map*, or *pseudo-morphism*, is an equivalence class of morphisms $U \rightarrow Y$, where $U \subseteq X$ is schematically dense; two such being equivalent if they agree on a schematically dense subset. If X, Y are S -preschemes, the equivalence class of an S -morphism $U \rightarrow Y$ is an *S -pseudo-morphism*.

The restriction of a pseudo-morphism $X \rightarrow Y$ to an open set $V \subseteq X$ makes sense. This gives a presheaf of sets on X , whose sections are pseudo-morphisms $V \rightarrow Y$.

An S -pseudo-morphism ω is *defined at* $x \in X$ if it is represented by an S -morphism on a neighborhood of x . The set of points where ω is defined is its *domain of definition* $\text{dom}_S(\omega)$ (or just $\text{dom}(\omega)$ if $S = \text{Spec}(\mathbb{Z})$).

Proposition (20.2.4). — *If X, Y are S -preschemes and Y is separated over S , then any pseudo-morphism $\omega: X \rightarrow Y$ is represented by an S -morphism $u: \text{dom}_S(\omega) \rightarrow Y$, and $\text{dom}_S(\omega)$ is (clearly) the maximal open set with this property.*

Corollary (20.2.5). — *If S is an S_0 -scheme and Y is an S -scheme, then for any S -pseudo-morphism $\omega: X \rightarrow Y$, we have $\text{dom}_S(\omega) = \text{dom}_{S_0}(\omega)$.*

Corollary (20.2.6). — *Under the hypotheses of (20.2.4), the presheaf of S -pseudo-morphisms to Y is a sheaf.*

Remark (20.2.7). — *If X is reduced, then U is schematically dense iff it is topologically dense. Then the concept of pseudo-morphism coincides with the classical notion of *rational map*.*

(20.2.8). A pseudo-morphism $X \rightarrow \text{Spec}(\mathbb{Z}[t])$, or equivalently, an X -pseudo-morphism $X \rightarrow X \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ is a *pseudo-function*. This amounts to an equivalence class of functions on schematically dense open sets, two such being equivalent if they agree on the intersection. For any pseudo-function ϕ , $\text{dom}(\phi)$ is the unique largest open set on which it is represented by a function.

By (20.2.6), the pseudo-functions form a sheaf of \mathcal{O}_X -algebras \mathcal{M}'_X . Define $M'(X) = \Gamma(X, \mathcal{M}'_X)$. A pseudo-function $\phi \in \mathcal{M}'_X(U)$ is *invertible* if it is represented by an invertible function on a schematically dense $U' \subseteq U$. The subsheaf of invertible pseudo functions can be identified with the sheaf of pseudo-morphisms to $\text{Spec}(\mathbb{Z}[t, t^{-1}])$.

Lemma (20.2.9). — *If $s \in \Gamma(U, \mathcal{O}_X)$ is regular, then $U_s = \{x \in U : s(x) \neq 0\}$ is schematically dense in U .*

(20.2.10). Let $f \in M(X) = \Gamma(X, \mathcal{M}_X)$ be a global meromorphic function. Then (20.2.9) implies that $\text{dom}(f)$ is schematically dense, so $f|_{\text{dom}(f)}$ defines a pseudo-function. Applying the same construction to open sets $V \subseteq X$ gives a canonical homomorphism of \mathcal{O}_X -algebras

$$(20.2.10.1) \quad \mathcal{M}_X \rightarrow \mathcal{M}'_X,$$

and an induced homomorphism of their sheaves of abelian groups

$$(20.2.10.2) \quad \mathcal{M}_X^* \rightarrow \mathcal{M}'_X^*.$$

Proposition (20.2.11). — *(i) The homomorphisms (20.2.10.1–2) are injective.*

(ii) If X is locally Noetherian, or if X is reduced and its set of irreducible components is locally finite, then (20.2.10.1–2) are bijective.

[Proof: since affines form a base of open sets, for (i) it suffices to show that if $X = \text{Spec}(A)$, $U \subseteq X$ schematically dense, $a, s \in A$ with s regular, and $(a/s)|_U = 0$, then $a/s = 0$. This is clear, since $(a/s)|_U = 0$ iff $(a/1)|_U = 0$, which implies $a|_U = 0$, since $\mathcal{O}_X \hookrightarrow \mathcal{M}_X$, and then $a = 0$, since U is schematically dense.

For (ii), it again suffices to consider $X \subseteq \text{Spec}(A)$, and to prove that every schematically dense $U \subseteq X$ contains an open set $D(s)$ for $s \in A$ regular—then a pseudo-function defined on U is defined on $D(s)$, hence represented by some $a/s^n \in M(X)$.

Either hypothesis in (ii) implies that there is a finite set of primes \mathfrak{p}_i in A such that (a) $A \setminus \bigcup_i \mathfrak{p}_i$ is the set of regular elements, and (b) each \mathfrak{p}_i is the annihilator of an element $a_i \in A$. If A is Noetherian, the set $\text{Ass}(A)$ has these properties, and if A is reduced with finitely many minimal primes, then the set of minimal primes does. If U is schematically dense, (b) implies $\mathfrak{p}_i \in U$ (otherwise $a_i|U = 0$). Hence if $U = X \setminus V(I)$, then $I \not\subseteq \bigcup_i \mathfrak{p}_i$, by prime avoidance, and there is a regular element $s \in I$ by (a). Then $D(s) \subseteq U$.]

Corollary (20.2.12). — *If $X = \text{Spec}(A)$ where A is Noetherian, or reduced with finitely many minimal primes, then $U \subseteq X$ is schematically dense iff there is a regular element $s \in A$ such that $D(s) \subseteq U$, and $M(X) = M'(X) = S^{-1}A$, where S is the set of regular elements of A .*

Remarks (20.2.13). — (i) The domain of definition of a meromorphic function $\phi \in M(X)$ is the same as that of its image in $M'(X)$.

(ii) If X is reduced, then \mathcal{M}'_X is the *sheaf of rational functions* \mathcal{R}_X ; which is then equal to \mathcal{M}_X if the set of irreducible components of X is locally finite. In general, there is a canonical homomorphism $\mathcal{M}'_X \rightarrow \mathcal{R}_X$, but it need not be injective, even if A is Noetherian.

(iii) Even if X is locally Noetherian, $\mathcal{M}'_X (= \mathcal{M}_X)$ is not quasi-coherent in general.

(iv) If X is locally Noetherian, then U is schematically dense if and only if $\text{Ass}(\mathcal{O}_X) \subseteq U$.

(20.2.14). Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module which is strictly torsion-free, thus $\mathcal{F} \subseteq \mathcal{M}_X(\mathcal{F})$. Given $u \in \Gamma(X, \mathcal{M}_X(\mathcal{F}))$, the annihilator $\mathcal{J} \subseteq \mathcal{O}_X$ of u in $\mathcal{M}_X(\mathcal{F})/\mathcal{F}$ is the *denominator ideal* of u . Then \mathcal{J} is quasi-coherent and the complement of $V(\mathcal{J})$ is the domain of definition of u .

Proposition (20.2.15). — *Given a morphism $f: X' \rightarrow X$, a quasi-coherent \mathcal{O}_X -module \mathcal{F} , and $\phi \in \Gamma(X, \mathcal{M}_f(\mathcal{F}))$, the set $f^{-1}(\text{dom}(\phi))$ is schematically dense in X' .*

Remark (20.2.16). — When Y is not separated, the presheaf of pseudo-morphisms into Y is in general not a sheaf.

§21: DIVISORS, 21.4, 6

21.3. Linear equivalence of divisors. (continued)

[Proof of (21.3.4): Under either hypothesis (a) or (b), if U is schematically dense, it follows from (20.2.11) that $\Gamma(X, \mathcal{M}_X(\mathcal{L})^*) = \Gamma(U, \mathcal{M}_X(\mathcal{L})^*)$. So it suffices to find a schematically dense U on which \mathcal{L} is trivial.

For (b), each generic point x of a component has a neighborhood U_x on which \mathcal{L} is trivial, and which is disjoint from the other components. Then U the union of these sets U_x suffices.

For (a), an affine U containing $\text{Ass}(\mathcal{O}_X)$ is schematically dense. Since U is affine and $\text{Ass}(\mathcal{O}_X)$ is finite, there exists a section $s \in \mathcal{L}(U)$ such that $\text{Ass}(\mathcal{O}_X) \subseteq U_s$. Then U_s is schematically dense, and $\mathcal{L}|_{U_s}$ is trivial, with generating section s .]

21.4. Inverse images of divisors.

(21.4.1). Given $f: X' \rightarrow X$, and (\mathcal{L}, s) corresponding to a divisor D , set $\mathcal{L}' = f^*(\mathcal{L})$. If $s \in \Gamma(X, \mathcal{M}_f(\mathcal{L}))$, then $s' = s \circ f$ exists, and if s' is regular, we get a well-defined divisor D' on X' corresponding to (\mathcal{L}', s') .

Definition (21.4.2). — We say that *the inverse image of D under f exists* if $s_D \in \Gamma(X, \mathcal{M}_f(\mathcal{O}_X(D)))$ and $s_{-D} \in \Gamma(X, \mathcal{M}_f(\mathcal{O}_X(-D)))$; then we define $f^*(D)$ to be the divisor corresponding to $(f^*(\mathcal{O}_X(D)), s_D \circ f)$.

The divisors whose inverse image exists form a subgroup $\text{Div}^f(X) \subseteq \text{Div}(X)$, and $f^*: \text{Div}^f(X) \rightarrow \text{Div}(X)$ is a homomorphism of ordered groups.

(21.4.3). If $f^*(D)$ exists, then the preimage of $D|U$ by $f^U: f^{-1}(U) \rightarrow U$ exists, for every open $U \subseteq X$, giving a well-defined subsheaf $\mathcal{D}iv_X^f \subseteq \mathcal{D}iv_X$. Let $\mathcal{M}_f^{**} \subseteq \mathcal{M}_f^*$ be the subsheaf of regular meromorphic functions whose inverse image exists and is again regular. Then $\mathcal{D}iv_X^f = \mathcal{M}_f^{**}/\mathcal{O}_X^*$ and $\text{Div}^f(X) = \Gamma(X, \mathcal{D}iv_X^f)$.

(21.4.4). Given another morphism $f': X'' \rightarrow X'$, if $f^*(D)$ and $f'^*(f^*(D))$ exist, then $(f \circ f')^*(D)$ exists and is equal to $f'^*(f^*(D))$.

Proposition (21.4.5). — *Any of the following conditions implies that $f^*(D)$ exists for every $D \in \text{Div}(X)$:*

(a) f is flat.

(b) X and X' are locally Noetherian preschemes and $f(\text{Ass}(\mathcal{O}_{X'})) \subseteq \text{Ass}(\mathcal{O}_X)$.

(c) X and X' are preschemes, the set of irreducible components of X is locally finite, X' is reduced, and each irreducible component of X' dominates an irreducible component of X .

Corollary (21.4.6). — *Let X be a prescheme which is locally Noetherian, or its set of irreducible components is locally finite. Then for every $x \in X$ there is a canonical isomorphism*

$$(21.4.6.1) \quad \mathcal{D}iv_{X,x} \cong \text{Div}(\mathcal{O}_{X,x}).$$

(21.4.7). If D is an effective divisor and $f^*(D)$ exists, then the closed subscheme $Y(f^*(D)) \subseteq X'$ is equal to $f^{-1}(Y(D))$ (21.2.12).

Proposition (21.4.8). — *If $f: X \rightarrow Y$ is a faithfully flat morphism of preschemes, then $f^*(D) \geq 0$ implies $D \geq 0$. In particular $D \mapsto f^*(D)$ is injective.*

21.6. Codimension-1 cycle associated to a divisor.

(21.6.1). Let X be a locally Noetherian prescheme. The set $\mathcal{I}(X)$ of irreducible closed subsets of X is in bijection with X by $x \mapsto \overline{\{x\}}$. Let $\mathfrak{Z}(X)$ be the subgroup of elements $(n_x)_{x \in X} \in \mathbb{Z}^X$ such that the set of irreducible closed subsets $\overline{\{x\}}$ for $n_x \neq 0$ is locally finite (if X is Noetherian, this is just the free abelian group $\mathbb{Z}^{(X)}$ with basis $\mathcal{I}(X)$). The elements of $\mathfrak{Z}(X)$ are *cycles*; those of $\mathcal{I}(X)$ are *prime cycles*. The group $\mathfrak{Z}(X)$ is ordered, with $(n_x) \in \mathfrak{Z}^+(X)$ if $n_x \geq 0$ for all x .

For $Z = (n_x)_{x \in X}$ we usually write

$$Z = \sum_x n_x \cdot \overline{\{x\}},$$

and call n_x the *multiplicity* $\text{mult}_x(Z)$. The *support* $\text{Supp}(Z)$ is the union of those $\overline{\{x\}}$ such that $\text{mult}_x(Z) \neq 0$. The local finiteness ensures that $\text{Supp}(Z)$ is closed. The dimension (resp. codimension) of Z is defined to be that of $\text{Supp}(Z)$.

(21.6.2). A closed subset $Y \subseteq X$ is *purely of codimension d* in X if every irreducible component of Y has codimension d . A codimension- d cycle is one whose support is purely of codimension d . Let $X^{(d)} = \{x \in X : \text{codim}(\overline{\{x\}}, X) = d\}$, (i.e., $\dim \mathcal{O}_{X,x} = d$). The codimension- d cycles form a subgroup $\mathfrak{Z}^d(X) \subseteq Z$, consisting of locally finite elements $Z = \sum_{x \in X^{(d)}} n_x \cdot \overline{\{x\}}$. Set $\mathfrak{Z}^{d+}(X) = \mathfrak{Z}^d(X) \cap \mathfrak{Z}^+(X)$. If X is Noetherian, $\mathfrak{Z}(X) = \bigoplus_d \mathfrak{Z}^d(X)$.

(21.6.3). If $Z = \sum n_x \overline{\{x\}}$, its *restriction* to an open set U is $Z|U = \sum_{x \in U} n_x \overline{\{x\}}$, a cycle on U with support $\text{Supp}(Z) \cap U$. Then $U \mapsto \mathfrak{Z}(U)$ (resp. $U \mapsto \mathfrak{Z}^d(U)$) is a presheaf \mathcal{Z}_X (resp. \mathcal{Z}_X^d) of abelian groups, and in fact a sheaf: namely, the direct sum over all $x \in X$ (resp. $x \in X^{(d)}$) of $(i_x)_*(\mathbb{Z})$. The positive cycles form a subsheaf $\mathcal{Z}^+(X)$ (resp. $\mathcal{Z}^{d+}(X)$). We always have $\mathcal{Z}_X = \bigoplus_d \mathcal{Z}_X^d$.

The sheaves $\mathcal{Z}_X^d, \mathcal{Z}_X$ are *flasque* [every section extends to a global section].

(21.6.4). We shall define a canonical homomorphism

$$(21.6.4.1) \quad c: \mathcal{D}iv_X \rightarrow \mathcal{Z}_X^1.$$

It suffices to define a homomorphism of sheaves of monoids $\mathcal{D}iv_X^+ \rightarrow \mathcal{Z}_X^1$, which will then extend uniquely to the desired c .

(21.6.5). To an effective divisor $D \in \text{Div}^+(X)$ corresponds the closed subscheme $Y(D)$, with ideal $\mathcal{I}_X(D) \subseteq \mathcal{O}_X$, it is regularly embedded of codimension 1 [i.e., $\mathcal{I}_X(D)$ is locally a principal ideal generated by a regular element]. Thus each generic point $x \in Y(D)$ has $\text{codim}(\overline{\{x\}}, X) = \dim(\mathcal{O}_{X,x}) = 1$, i.e., $x \in X^{(1)}$. The set of these points x is locally finite, and $\mathcal{O}_{Y(D),x}$ is Artinian. If $x \in X^{(1)}$ is not a generic point of $Y(D)$, then $x \notin Y(D)$, and $\mathcal{O}_{Y(D),x} = 0$. Set

$$\text{cyc}(D) = \sum_{x \in X^{(1)}} l(\mathcal{O}_{Y(D),x}) \cdot \overline{\{x\}} \in \mathfrak{Z}^1(X).$$

Proposition (21.6.6). — $D \mapsto \text{cyc}(D)$ is a monoid homomorphism $\text{Div}^+(X) \rightarrow \mathfrak{Z}^{1+}(X)$.

It follows that cyc extends uniquely to the desired sheaf homomorphism (21.6.4.1).

We have $\text{Supp}(\text{cyc}(D)) \subseteq \text{Supp}(D)$ with equality if $D \geq 0$.

(21.6.7). Define $\text{mult}_x(D) = \text{mult}_x(\text{cyc}(D))$; this integer is called the multiplicity of D at x , or of the prime cycle $\overline{\{x\}}$ in D . If f is a meromorphic function, define $o_x(f) = \text{mult}_x(\text{div}(f))$, called the *order* of f at x . The codimension-1 cycles

$$Z^+(f) = \sum_{x \in X^{(1)}} o_x(f)^+ \cdot \overline{\{x\}}, \quad Z^-(f) = \sum_{x \in X^{(1)}} o_x(f)^- \cdot \overline{\{x\}}$$

are the *cycle of zeroes* and the *cycle of poles* of f . We have $\text{cyc}(\text{div}(f)) = Z^+(f) - Z^-(f)$. Cycles of the form $\text{cyc}(\text{div}(f))$ are *principal*, or *linearly equivalent to zero*. They form a subgroup $\mathfrak{Z}_{\text{princ}}^1(X) \subseteq \mathfrak{Z}^1(X)$.

Elements of $\Gamma(X, c(\mathcal{D}iv_X)) \subseteq \mathfrak{Z}^1(X)$ are called *locally principle cycles*. The group $\mathfrak{C}\mathfrak{l}(X) = \mathfrak{Z}^1(X)/\mathfrak{Z}_{\text{princ}}^1(X)$ is the *cycle class group*. There is a canonical homomorphism

$$\text{Div}(X)/\text{Div}_{\text{princ}}(X) \rightarrow \mathfrak{C}\mathfrak{l}(X).$$

(21.6.8). Suppose $X = \text{Spec}(A)$ where A is an *integrally closed domain*. Then $X^{(1)}$ is the set of *height 1 primes* of A , and $\mathfrak{C}\mathfrak{l}(X)$ is the *divisor class group* of A .

Theorem (21.6.9). — *Let X be a locally Noetherian and normal prescheme.*

(i) $\text{cyc}: \text{Div}(X) \rightarrow \mathfrak{Z}^1(X)$ is injective, with image consisting of the *locally principal divisors*.

(ii) *The following are equivalent:*

(a) $\text{cyc}: \text{Div}(X) \rightarrow \mathfrak{Z}^1(X)$ is bijective.

(b) *Every codimension-1 cycle is locally principal.*

(c) *For all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is factorial [i.e., a UFD]. (Then X is said to be locally factorial).*

If A is factorial, then $X = \text{Spec}(A)$ is locally factorial. Writing $0 \neq f \in K(A)$ as r/s with r, s relatively prime, the divisors $\text{div}(r), \text{div}(s)$ are determined and are the divisors of zeroes and poles, respectively, of f (21.6.7).

Corollary (21.6.10). — *Let X be a locally Noetherian and normal prescheme.*

(i) *There is a canonical injective homomorphism*

$$(21.6.10.1) \quad \text{Pic}(X) \rightarrow \mathfrak{C}\mathfrak{l}(X).$$

(ii) *If X is locally factorial, (21.6.10.1) is bijective, and conversely.*

Corollary (21.6.11). — *Let X be a locally Noetherian and locally factorial prescheme. Then the sheaf $\mathcal{D}iv_X$ is flasque, and for every open $U \subseteq X$, the canonical homomorphism $\text{Pic}(X) \rightarrow \text{Pic}(U)$ is surjective.*

Proposition (21.6.12). — *Let X be a reduced, Noetherian prescheme. Let $(U_\lambda)_{\lambda \in L}$ be a decreasingly filtered system of open subsets of X such that*

(1) *If $Y_\lambda = X \setminus U_\lambda$, then $\text{codim}(Y_\lambda, X) \geq 2$, for all $\lambda \in L$.*

(2) *For every $x \in \bigcap_{\lambda \in L} U_\lambda$, $\mathcal{O}_{X,x}$ is factorial.*

Then there are canonical isomorphisms $\varinjlim \text{Div}(U_\lambda) \cong \mathfrak{Z}^1(X)$, $\varinjlim \text{Pic}(U_\lambda) \cong \mathfrak{C}\mathfrak{l}(X)$, commuting with restriction to open subsets $V \subseteq X$.

Corollary (21.6.13). — *Let A be a Noetherian local integrally closed domain of dimension ≥ 2 , $X = \text{Spec}(A)$, \mathfrak{a} the closed point, $U = X \setminus \{\mathfrak{a}\}$. Then A is factorial iff U is locally factorial and $\text{Pic}(U) = 0$.*

Corollary (21.6.14). — Let A be a local Noetherian ring of dimension ≥ 2 , $X = \text{Spec}(A)$, \mathfrak{a} the closed point, $U = X \setminus \{\mathfrak{a}\}$. Then A is factorial if and only if U is locally factorial, $\text{Pic}(U) = 0$, and $\text{depth}(A) \geq 2$.