

SYNOPSIS OF MATERIAL FROM EGA IV

§20: MEROMORPHIC AND PSEUDO-MEROMORPHIC FUNCTIONS, 20.1.1–10

§21: DIVISORS, 21.1–3

20.1. **Meromorphic functions.** [Liu, 7.1.12–16, but 7.1.12(a) is not quite correct.]

(20.1.1–10) An element  $x$  in a ring  $A$  is *regular* [or a non-zero-divisor, or NZD] if  $\text{ann}(x) = 0$ . If  $(X, \mathcal{O}_X)$  is a ringed space, a section  $s \in \mathcal{O}_X(U)$  is regular iff  $s_x \in \mathcal{O}_{X,x}$  is regular for all  $x \in U$ . The regular elements form a subsheaf of sets  $\mathcal{S} \subseteq \mathcal{O}_X$ . The sheaf of  $\mathcal{O}_X$ -algebras associated to the presheaf  $U \mapsto \mathcal{O}_X(U)[\mathcal{S}_X(U)^{-1}]$  is denoted

$$(20.1.0.1) \quad \mathcal{M}_X = \mathcal{O}_X[\mathcal{S}^{-1}]$$

and called the sheaf of *meromorphic functions*. We have  $\mathcal{O}_X \subseteq \mathcal{M}_X$ . A meromorphic function  $\phi \in M(X) = \Gamma(X, \mathcal{M}_X)$  is *defined* on  $U$  if  $\phi|_U \in \mathcal{O}_X(U)$ . There is a largest such open set  $U$ , called the *domain of definition* of  $\phi$ .

If  $X$  is a reduced prescheme, then for each maximal point  $\xi \in X$ , *i.e.*, each generic point of a component of  $X$ ,  $\mathcal{O}_{X,\xi}$  is a field, and  $s \in \Gamma(U, \mathcal{O}_X)$  is an NZD iff  $s_\xi \neq 0$  for each maximal point  $\xi \in U$ .

[If  $X$  is integral, then  $\mathcal{M}_X$  is the constant sheaf  $\mathcal{M}_X(U) = K$  for all non-empty  $U$ , where  $K = \mathcal{O}_{X,\xi}$  is the residue field at the generic point  $\xi$ , the *field of rational functions* on  $X$ .]

Given an  $\mathcal{O}_X$ -module  $\mathcal{F}$ , the sheaf

$$(20.1.0.2) \quad \mathcal{M}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X = \mathcal{F}[\mathcal{S}^{-1}]$$

is the *sheaf of meromorphic sections of  $\mathcal{F}$* . There is a canonical homomorphism  $\mathcal{F} \rightarrow \mathcal{M}(\mathcal{F})$ . If it is injective,  $\mathcal{F}$  is called *strictly torsion-free*; this means every section of  $\mathcal{S}$  acts injectively on  $\mathcal{F}$ . In particular, a locally free sheaf is strictly torsion-free. If  $X$  is a locally Noetherian prescheme and  $\mathcal{F}$  is quasi-coherent, then  $\mathcal{F}$  is strictly torsion free iff  $\text{Ass}(\mathcal{F}) \subseteq \text{Ass}(\mathcal{O}_X)$  [Liu, 7.1.1]. A section  $s \in \mathcal{M}_X(\mathcal{F})$  is *defined* at  $u$  if its restriction to some neighborhood  $V \ni u$  is in the image of  $\mathcal{F}(V) \rightarrow (\mathcal{M}_X(\mathcal{F}))(V)$ . The *domain of definition* of  $s$  is the largest  $U$  such that  $s$  is defined at all  $u \in U$ .

Denote by  $\mathcal{M}_X^*$  the sheaf of groups of units in  $\mathcal{M}_X$ . It is the sheaf of regular elements in  $\mathcal{M}_X$ ; its sections are called *regular meromorphic functions*.

If  $\mathcal{L}$  is an invertible  $\mathcal{O}_X$ -module, then  $\mathcal{M}_X(\mathcal{L})$  is an invertible  $\mathcal{M}_X$ -module. The condition that a section of  $\mathcal{M}_X(\mathcal{L})$  correspond to an invertible element under a local isomorphism  $\mathcal{M}_X(\mathcal{L})|_U \cong \mathcal{M}_X|_U$  is independent of the chosen local isomorphism. Such sections are the *regular meromorphic sections* of  $\mathcal{M}(\mathcal{L})$ ; they form a subsheaf of sets  $\mathcal{M}_X(\mathcal{L})^*$ . If  $s \in \Gamma(X, \mathcal{M}_X(\mathcal{L})^*)$ , then  $t \mapsto ts$  is an isomorphism of  $M(X)$  on  $M(X, \mathcal{L}) = \Gamma(X, \mathcal{M}_X(\mathcal{L}))$ . The regular sections of  $\mathcal{M}_X(\mathcal{L})$  are those  $s$  for which there exists  $s^{-1} \in \mathcal{M}_X(\mathcal{L}^{-1})$  such that  $s \otimes s^{-1}$  is the unit section of  $\mathcal{M}_X$ .

§21: DIVISORS

21.1. **Divisors on a ringed space.**

(21.1.1). Let  $(X, \mathcal{O}_X)$  be a ringed space,  $\mathcal{M}_X, \mathcal{M}_X^*$  as in (20.1).

*Definition (21.1.2).* — The sheaf of [Cartier] divisors on  $X$  is  $\mathcal{D}iv_X = \mathcal{M}_X^*/\mathcal{O}_X^*$ . We put  $\text{Div}(X) = \Gamma(X, \mathcal{D}iv_X)$ . The image in  $\text{Div}(X)$  of a regular meromorphic function  $f \in \Gamma(X, \mathcal{M}_X^*)$  is the *divisor of  $f$* , denoted  $\text{div}(f)$  [Liu, 7.1.17].

The *support* of a divisor  $D$  is  $\text{Supp}(D) = \{x \in X : D_x \neq 0\}$ .

Clearly  $\mathcal{D}iv_X|_U = \mathcal{D}iv_U$ , hence  $\mathcal{D}iv_X(U) = \text{Div}(U)$ .

(21.1.3). The group  $\text{Div}(X)$  is written additively; thus  $\text{div}(fg) = \text{div}(f) + \text{div}(g)$ ,  $\text{div}(f^{-1}) = -\text{div}(f)$ . By definition, if  $f \in \Gamma(X, \mathcal{O}_X^*)$  is an everywhere-defined regular meromorphic function then  $\text{div}(f) = 0$ ; and  $\text{div}(f) = \text{div}(g)$  iff  $fg^{-1} \in \Gamma(X, \mathcal{O}_X^*)$ .

(21.1.4). Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. To any regular meromorphic section  $s \in M(X, \mathcal{L})$  there corresponds a well-defined divisor  $\text{div}(s)$  such that on any open set  $U$  where  $\mathcal{L}|_U \cong \mathcal{O}_U$ ,  $\text{div}(s)$  coincides with  $\text{div}(f)$ , where  $f \in \mathcal{M}_X(U)$  corresponds to  $s \in (\mathcal{M}_X(\mathcal{L}))(U)$  (this does not depend on the choice of isomorphism  $\mathcal{L}|_U \cong \mathcal{O}_U$ ).

We have  $\text{div}(s \otimes s') = \text{div}(s) + \text{div}(s')$ , and for  $s, s'$  belonging to the same  $\mathcal{L}$ , we have  $\text{div}(s) = \text{div}(s')$  iff  $s/s' \in \Gamma(X, \mathcal{O}_X^*)$ .

(21.1.5). The sheaf of regular elements  $\mathcal{S}(\mathcal{O}_X) = \mathcal{O}_X \cap \mathcal{M}_X^*$  is a sub-sheaf of monoids in  $\mathcal{M}_X^*$ , and we have  $\mathcal{S}(\mathcal{O}_X) \cap \mathcal{S}(\mathcal{O}_X)^{-1} = \mathcal{O}_X^*$ .

*Definition (21.1.6).* — The image of  $\mathcal{S}(\mathcal{O})$  in  $\mathcal{D}iv_X$  is the sheaf of *effective divisors*, denoted  $\mathcal{D}iv^+(X)$ ; we put  $\text{Div}^+(X) = \Gamma(X, \mathcal{D}iv^+(X))$ .

We have

$$(21.1.6.1) \quad \text{Div}^+(X) + \text{Div}^+(X) \subseteq \text{Div}^+(X),$$

$$(21.1.6.2) \quad \text{Div}^+(X) \cap (-\text{Div}^+(X)) = \{0\},$$

so  $\text{Div}^+(X)$  is the set of positive elements for a partial order on the group of divisors,  $D \leq D'$  iff  $D' - D \in \text{Div}^+(X)$ . This makes  $\mathcal{D}iv(X)$  a sheaf of partially ordered groups. Note that  $\text{div}(f) \geq 0$  means  $f \in \Gamma(X, \mathcal{S}(\mathcal{O}_X))$ , *i.e.*,  $f$  is a global function invertible in  $\mathcal{M}_X$ .

(21.1.7). If  $s$  is a regular meromorphic section of  $\mathcal{L}$ , then  $\text{div}(s) \geq 0$  iff  $s \in \Gamma(X, \mathcal{L}) \cap \Gamma(X, \mathcal{M}_X(\mathcal{L})^*)$ .

*Proposition (21.1.8).* — *Let  $D$  be a divisor on a locally Noetherian prescheme  $X$ . Suppose that for all  $x \in X$  such that  $\text{depth}(\mathcal{O}_{X,x}) = 1$ , we have  $D_x \geq 0$  (resp.  $D_x = 0$ ). Then  $D \geq 0$  (resp.  $D = 0$ ).*

[The *depth* of Noetherian local ring  $(A, \mathfrak{m})$  is the maximum length of a *regular sequence*  $x_1, \dots, x_d \in \mathfrak{m}$  such that  $x_i$  is an NZD in  $A/(x_1, \dots, x_{i-1})$  for all  $i$ . The proof is based on the fact that if  $U = X \setminus Z$  is open, and  $\text{depth}(\mathcal{O}_{X,z}) \geq 2$  for all  $z \in Z$ , then  $\Gamma(X, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X)$ .

If  $X$  is *normal* (*i.e.*  $X$  can be covered by affines  $U = \text{Spec}(A)$  where  $A$  is integrally closed), then  $\text{depth}(\mathcal{O}_{X,x}) = 1$  iff  $V(x)$  has codimension 1 in  $X$ .]

*Corollary (21.1.9).* — *If  $X$  is locally Noetherian and  $D$  is a divisor, then every maximal point  $x \in \text{Supp}(D)$  has  $\text{depth}(\mathcal{O}_{X,x}) = 1$ .*

*Proposition (21.1.10).* — *If  $(A, \mathfrak{m})$  is a Noetherian local ring, we have  $\text{Div}(A) = 0$  iff  $\text{depth}(A) = 0$ , that is, iff  $\mathfrak{m} \in \text{Ass}(A)$ .*

## 21.2. Divisors and invertible fractional ideals.

(21.2.1). A sub- $\mathcal{O}_X$ -module  $\mathcal{J} \subseteq \mathcal{M}_X$  is called a *fractional ideal*; it is *invertible* if  $\mathcal{J}$  is an invertible  $\mathcal{O}_X$ -module.

*Proposition (21.2.2).* — *A fractional ideal  $\mathcal{J}$  is invertible iff every  $x \in X$  has a neighborhood  $U$  such that  $\mathcal{J}|_U = \mathcal{O}_U f$  for some  $f \in \Gamma(U, \mathcal{M}_X^*)$ .*

The section  $f$  in the proposition is unique up to multiplication by an element of  $\Gamma(U, \mathcal{O}_X^*)$ .

*Corollary (21.2.3).* — *(i) Let  $\mathcal{J}$  be an invertible fractional ideal, and define  $\mathcal{J}'$  by  $\mathcal{J}'|_U = \mathcal{O}_U f^{-1}$  for every  $U$  such that  $\mathcal{J}|_U = \mathcal{O}_U f$ . Then  $\mathcal{J}^{-1} \cong \mathcal{J}'$ .*

*(ii) If  $\mathcal{J}_1, \mathcal{J}_2$  are invertible fractional ideals, then  $\mathcal{J}_1 \otimes \mathcal{J}_2 \cong \mathcal{J}_1 \mathcal{J}_2$ .*

(21.2.4). It follows that the set  $\text{Idinv}(X)$  of fractional ideals forms a group under multiplication, with identity  $\mathcal{O}_X$ . The presheaf  $U \mapsto \text{Idinv}(U)$  is a sheaf of abelian groups  $\mathcal{I}dinv_X$ .

(21.2.5). The map  $f \mapsto \mathcal{J}(f) = \mathcal{O}_X f$  is a group homomorphism from  $\Gamma(X, \mathcal{M}_X^*)$  to  $\text{Idinv}(X)$ . Considering this on open sets, we get a sheaf homomorphism

$$(21.2.5.1) \quad I_0: \mathcal{M}_X^* \rightarrow \mathcal{I}dinv_X$$

If  $f \in \Gamma(X, \mathcal{O}_X^*)$  then  $\mathcal{J}(f) = \mathcal{O}_X$ , hence  $I_0$  factors as

$$(21.2.5.2) \quad \mathcal{M}_X^* \rightarrow \mathcal{M}_X^*/\mathcal{O}_X^* = \mathcal{D}iv_X \xrightarrow{I} \mathcal{I}dinv_X.$$

In particular, we have a homomorphism  $\mathcal{I}_U: \text{Div}(U) \rightarrow \text{Idinv}(U)$  such that  $\mathcal{I}_U(\text{div}(f)) = \mathcal{O}_U f$ , and for any divisor  $D$ , we have  $f \in \Gamma(X, \mathcal{I}_X(D))$  iff  $\text{div}(f) \geq D$ .

*Proposition (21.2.6).* — *The homomorphism  $I: \mathcal{D}iv_X \rightarrow \mathcal{I}dinv_X$  is bijective.*

(21.2.7). We identify  $\mathcal{D}iv_X$  and  $\mathcal{I}dinv_X$ . For  $D \in \text{Div}(X)$ , we have  $D \geq 0$  iff  $\mathcal{I}_X(D) \subseteq \mathcal{O}_X$ , so  $\text{Div}^+(X)$  corresponds to the set of ideal sheaves  $\mathcal{J} \subseteq \mathcal{O}_X$  such that  $\mathcal{J}$  is an invertible  $\mathcal{O}_X$ -module. [If  $X$  is integral, this amounts to  $\mathcal{J}_X$  being locally a principal ideal.] More generally,  $D_1 \geq D_2$  iff  $\mathcal{I}_X(D_1) \subseteq \mathcal{I}_X(D_2)$ .

(21.2.8). We define  $\mathcal{O}_X(D) = \mathcal{I}_X(D)^{-1}$ .

(21.2.9). For any fractional ideal  $\mathcal{J}$ , the inclusion  $\mathcal{J} \subseteq \mathcal{M}_X$  induces

$$(21.2.9.1) \quad \mathcal{M}_X(\mathcal{J}) = \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{M}_X \rightarrow \mathcal{M}_X \otimes_{\mathcal{O}_X} \mathcal{M}_X = \mathcal{M}_X.$$

In particular for any divisor  $D$  we have a canonical isomorphism

$$(21.2.9.2) \quad \mathcal{M}_X(\mathcal{O}_X(D)) \cong \mathcal{M}_X$$

and we denote by  $s_D$  the regular meromorphic section of  $\mathcal{O}_X(D)$  corresponding to  $1 \in \mathcal{M}_X$ . Then

$$(21.2.9.3) \quad \text{div}(s_D) = D,$$

and we have

$$(21.2.9.4) \quad s_0 = 1, \quad s_{D+D'} = s_D \otimes s_{D'}, \quad s_{nD} = s_D^{\otimes n} \quad (n \in \mathbb{Z}).$$

(21.2.10). Let  $s, s'$  be regular meromorphic sections of invertible sheaves  $\mathcal{L}, \mathcal{L}'$ . Call  $(\mathcal{L}, s), (\mathcal{L}', s')$  *equivalent* if there is an isomorphism  $u: \mathcal{L} \rightarrow \mathcal{L}'$  carrying  $s$  to  $s'$ . Denote an equivalence class by  $\text{cl}(\mathcal{L}, s)$  and let  $D(X)$  denote the set of equivalence classes. Tensor product makes  $D(X)$  an abelian group.

*Proposition (21.2.11).* — *The groups  $\text{Div}(X)$  and  $D(X)$  are isomorphic, via*

$$(21.2.11.1) \quad D \rightarrow \text{cl}(\mathcal{O}_X(D), s_D), \quad \text{cl}(\mathcal{L}, s) \rightarrow \text{div}(s).$$

(21.2.12). If  $D \geq 0$  is an effective divisor on a prescheme  $X$ , then  $\mathcal{I}_X(D)$  is an ideal sheaf in  $\mathcal{O}_X$ , defining a closed subscheme  $Y(D) \subseteq X$ . Locally we have  $\mathcal{I}_X(D) = \mathcal{O}_U t$  for some regular section  $t \in \Gamma(U, \mathcal{O})$ , that is, the immersion  $Y(D) \hookrightarrow X$  is *regular of codimension 1* at every point. We have  $\text{Supp}(D) = Y(D)$ . Conversely, every closed subscheme regular of codimension 1 is  $Y(D)$  for an effective divisor  $D$ .

### 21.3. Linear equivalence of divisors.

(21.3.1). A divisor  $D = \text{div}(f)$  of  $f \in \Gamma(X, \mathcal{M}_X^*)$  is said to be *principal*. The principal divisors form a subgroup  $\Gamma(X, \mathcal{M}_X^*)/\Gamma(X, \mathcal{O}_X^*)$  of  $\text{Div}(X)$ . If  $D - D'$  is principal,  $D$  and  $D'$  are *linearly equivalent* [Liu, 7.1.17].

(21.3.2). Let  $\text{Pic}(X)$  be the group of isomorphism classes of invertible  $\mathcal{O}_X$ -modules, under tensor product, writing  $\text{cl}(\mathcal{L})$  for the isomorphism class of  $\mathcal{L}$ . Then

$$l_X: \text{cl}(\mathcal{L}, s) \mapsto \text{cl}(\mathcal{L})$$

is a homomorphism  $\text{Div}(X) \cong D(X) \rightarrow \text{Pic}(X)$ , and  $l_X(D) = \text{cl}(\mathcal{O}_X(D))$ .

Given  $u: X' \rightarrow X$ , we have a homomorphism  $\text{Pic}(u): \text{Pic}(X) \rightarrow \text{Pic}(X')$  given by  $u^*$ .

*Proposition (21.3.3).* — (i) *The kernel of  $l_X: \text{Div}(X) \rightarrow \text{Pic}(X)$  consists of the principal divisors, i.e.,  $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$  iff  $D$  is linearly equivalent to  $D'$ .*

(ii) *An invertible sheaf  $\mathcal{L}$  is of the form  $l_X(D)$  iff there exists a regular meromorphic section of  $\mathcal{L}$ .*

*Proposition (21.3.4).* — *Let  $X$  be a prescheme satisfying either of:*

(a)  *$X$  is locally Noetherian and  $\text{Ass}(X)$  is contained in some open affine  $U \subseteq X$ .*

(b)  *$X$  is reduced and its set of irreducible components is locally finite.*

*Then  $l_X: \text{Div}(X) \rightarrow \text{Pic}(X)$  is surjective, i.e.,  $\text{Div}(X)/\text{Div.princ}(X) \cong \text{Pic}(X)$  [Liu, 7.1.19–20].*

*Corollary (21.3.5).* — *If  $X$  is a Noetherian prescheme on which there exists an ample sheaf (e.g., a quasi-projective scheme over  $\text{Spec}$  of a Noetherian ring), then  $l_X: \text{Div}(X) \rightarrow \text{Pic}(X)$  is surjective.*

[Proof:  $\text{Ass}(X)$  is finite; apply (II, 4.5.4).]