

SYNOPSIS OF MATERIAL FROM EGA IV

§20: MEROMORPHIC AND PSEUDO-MEROMORPHIC FUNCTIONS, 20.1.1–10

§21: DIVISORS, 21.1–3

20.1. **Meromorphic functions.** [Liu, 7.1.12–16, but 7.1.12(a) is not quite correct.]

(20.1.1–10) An element x in a ring A is *regular* [or a non-zero-divisor, or NZD] if $\text{ann}(x) = 0$. If (X, \mathcal{O}_X) is a ringed space, a section $s \in \mathcal{O}_X(U)$ is regular iff $s_x \in \mathcal{O}_{X,x}$ is regular for all $x \in U$. The regular elements form a subsheaf of sets $\mathcal{S} \subseteq \mathcal{O}_X$. The sheaf of \mathcal{O}_X -algebras associated to the presheaf $U \mapsto \mathcal{O}_X(U)[\mathcal{S}_X(U)^{-1}]$ is denoted

$$(20.1.0.1) \quad \mathcal{M}_X = \mathcal{O}_X[\mathcal{S}^{-1}]$$

and called the sheaf of *meromorphic functions*. We have $\mathcal{O}_X \subseteq \mathcal{M}_X$. A meromorphic function $\phi \in M(X) = \Gamma(X, \mathcal{M}_X)$ is *defined* on U if $\phi|_U \in \mathcal{O}_X(U)$. There is a largest such open set U , called the *domain of definition* of ϕ .

If X is a reduced prescheme, then for each maximal point $\xi \in X$, *i.e.*, each generic point of a component of X , $\mathcal{O}_{X,\xi}$ is a field, and $s \in \Gamma(U, \mathcal{O}_X)$ is an NZD iff $s_\xi \neq 0$ for each maximal point $\xi \in U$.

[If X is integral, then \mathcal{M}_X is the constant sheaf $\mathcal{M}_X(U) = K$ for all non-empty U , where $K = \mathcal{O}_{X,\xi}$ is the residue field at the generic point ξ , the *field of rational functions* on X .]

Given an \mathcal{O}_X -module \mathcal{F} , the sheaf

$$(20.1.0.2) \quad \mathcal{M}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X = \mathcal{F}[\mathcal{S}^{-1}]$$

is the *sheaf of meromorphic sections of \mathcal{F}* . There is a canonical homomorphism $\mathcal{F} \rightarrow \mathcal{M}(\mathcal{F})$. If it is injective, \mathcal{F} is called *strictly torsion-free*; this means every section of \mathcal{S} acts injectively on \mathcal{F} . In particular, a locally free sheaf is strictly torsion-free. If X is a locally Noetherian prescheme and \mathcal{F} is quasi-coherent, then \mathcal{F} is strictly torsion free iff $\text{Ass}(\mathcal{F}) \subseteq \text{Ass}(\mathcal{O}_X)$ [Liu, 7.1.1]. A section $s \in \mathcal{M}_X(\mathcal{F})$ is *defined* at u if its restriction to some neighborhood $V \ni u$ is in the image of $\mathcal{F}(V) \rightarrow (\mathcal{M}_X(\mathcal{F}))(V)$. The *domain of definition* of s is the largest U such that s is defined at all $u \in U$.

Denote by \mathcal{M}_X^* the sheaf of groups of units in \mathcal{M}_X . It is the sheaf of regular elements in \mathcal{M}_X ; its sections are called *regular meromorphic functions*.

If \mathcal{L} is an invertible \mathcal{O}_X -module, then $\mathcal{M}_X(\mathcal{L})$ is an invertible \mathcal{M}_X -module. The condition that a section of $\mathcal{M}_X(\mathcal{L})$ correspond to an invertible element under a local isomorphism $\mathcal{M}_X(\mathcal{L})|_U \cong \mathcal{M}_X|_U$ is independent of the chosen local isomorphism. Such sections are the *regular meromorphic sections* of $\mathcal{M}(\mathcal{L})$; they form a subsheaf of sets $\mathcal{M}_X(\mathcal{L})^*$. If $s \in \Gamma(X, \mathcal{M}_X(\mathcal{L})^*)$, then $t \mapsto ts$ is an isomorphism of $M(X)$ on $M(X, \mathcal{L}) = \Gamma(X, \mathcal{M}_X(\mathcal{L}))$. The regular sections of $\mathcal{M}_X(\mathcal{L})$ are those s for which there exists $s^{-1} \in \mathcal{M}_X(\mathcal{L}^{-1})$ such that $s \otimes s^{-1}$ is the unit section of \mathcal{M}_X .

§21: DIVISORS

21.1. **Divisors on a ringed space.**

(21.1.1). Let (X, \mathcal{O}_X) be a ringed space, $\mathcal{M}_X, \mathcal{M}_X^*$ as in (20.1).

Definition (21.1.2). — The sheaf of [Cartier] divisors on X is $\mathcal{D}iv_X = \mathcal{M}_X^*/\mathcal{O}_X^*$. We put $\text{Div}(X) = \Gamma(X, \mathcal{D}iv_X)$. The image in $\text{Div}(X)$ of a regular meromorphic function $f \in \Gamma(X, \mathcal{M}_X^*)$ is the *divisor of f* , denoted $\text{div}(f)$ [Liu, 7.1.17].

The *support* of a divisor D is $\text{Supp}(D) = \{x \in X : D_x \neq 0\}$.

Clearly $\mathcal{D}iv_X|_U = \mathcal{D}iv_U$, hence $\mathcal{D}iv_X(U) = \text{Div}(U)$.

(21.1.3). The group $\text{Div}(X)$ is written additively; thus $\text{div}(fg) = \text{div}(f) + \text{div}(g)$, $\text{div}(f^{-1}) = -\text{div}(f)$. By definition, if $f \in \Gamma(X, \mathcal{O}_X^*)$ is an everywhere-defined regular meromorphic function then $\text{div}(f) = 0$; and $\text{div}(f) = \text{div}(g)$ iff $fg^{-1} \in \Gamma(X, \mathcal{O}_X^*)$.

(21.1.4). Let \mathcal{L} be an invertible \mathcal{O}_X -module. To any regular meromorphic section $s \in M(X, \mathcal{L})$ there corresponds a well-defined divisor $\text{div}(s)$ such that on any open set U where $\mathcal{L}|_U \cong \mathcal{O}_U$, $\text{div}(s)$ coincides with $\text{div}(f)$, where $f \in \mathcal{M}_X(U)$ corresponds to $s \in (\mathcal{M}_X(\mathcal{L}))(U)$ (this does not depend on the choice of isomorphism $\mathcal{L}|_U \cong \mathcal{O}_U$).

We have $\text{div}(s \otimes s') = \text{div}(s) + \text{div}(s')$, and for s, s' belonging to the same \mathcal{L} , we have $\text{div}(s) = \text{div}(s')$ iff $s/s' \in \Gamma(X, \mathcal{O}_X^*)$.

(21.1.5). The sheaf of regular elements $\mathcal{S}(\mathcal{O}_X) = \mathcal{O}_X \cap \mathcal{M}_X^*$ is a sub-sheaf of monoids in \mathcal{M}_X^* , and we have $\mathcal{S}(\mathcal{O}_X) \cap \mathcal{S}(\mathcal{O}_X)^{-1} = \mathcal{O}_X^*$.

Definition (21.1.6). — The image of $\mathcal{S}(\mathcal{O})$ in $\mathcal{D}iv_X$ is the sheaf of *effective divisors*, denoted $\mathcal{D}iv^+(X)$; we put $\text{Div}^+(X) = \Gamma(X, \mathcal{D}iv^+(X))$.

We have

$$(21.1.6.1) \quad \text{Div}^+(X) + \text{Div}^+(X) \subseteq \text{Div}^+(X),$$

$$(21.1.6.2) \quad \text{Div}^+(X) \cap (-\text{Div}^+(X)) = \{0\},$$

so $\text{Div}^+(X)$ is the set of positive elements for a partial order on the group of divisors, $D \leq D'$ iff $D' - D \in \text{Div}^+(X)$. This makes $\mathcal{D}iv(X)$ a sheaf of partially ordered groups. Note that $\text{div}(f) \geq 0$ means $f \in \Gamma(X, \mathcal{S}(\mathcal{O}_X))$, *i.e.*, f is a global function invertible in \mathcal{M}_X .

(21.1.7). If s is a regular meromorphic section of \mathcal{L} , then $\text{div}(s) \geq 0$ iff $s \in \Gamma(X, \mathcal{L}) \cap \Gamma(X, \mathcal{M}_X(\mathcal{L})^*)$.

Proposition (21.1.8). — Let D be a divisor on a locally Noetherian prescheme X . Suppose that for all $x \in X$ such that $\text{depth}(\mathcal{O}_{X,x}) = 1$, we have $D_x \geq 0$ (resp. $D_x = 0$). Then $D \geq 0$ (resp. $D = 0$).

[The *depth* of Noetherian local ring (A, \mathfrak{m}) is the maximum length of a *regular sequence* $x_1, \dots, x_d \in \mathfrak{m}$ such that x_i is an NZD in $A/(x_1, \dots, x_{i-1})$ for all i . The proof is based on the fact that if $U = X \setminus Z$ is open, and $\text{depth}(\mathcal{O}_{X,z}) \geq 2$ for all $z \in Z$, then $\Gamma(X, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X)$.

If X is *normal* (*i.e.* X can be covered by affines $U = \text{Spec}(A)$ where A is integrally closed), then $\text{depth}(\mathcal{O}_{X,x}) = 1$ iff $V(x)$ has codimension 1 in X .]

Corollary (21.1.9). — If X is locally Noetherian and D is a divisor, then every maximal point $x \in \text{Supp}(D)$ has $\text{depth}(\mathcal{O}_{X,x}) = 1$.

Proposition (21.1.10). — If (A, \mathfrak{m}) is a Noetherian local ring, we have $\text{Div}(A) = 0$ iff $\text{depth}(A) = 0$, that is, iff $\mathfrak{m} \in \text{Ass}(A)$.

21.2. Divisors and invertible fractional ideals.

(21.2.1). A sub- \mathcal{O}_X -module $\mathcal{J} \subseteq \mathcal{M}_X$ is called a *fractional ideal*; it is *invertible* if \mathcal{J} is an invertible \mathcal{O}_X -module.

Proposition (21.2.2). — *A fractional ideal \mathcal{J} is invertible iff every $x \in X$ has a neighborhood U such that $\mathcal{J}|_U = \mathcal{O}_U f$ for some $f \in \Gamma(U, \mathcal{M}_X^*)$.*

The section f in the proposition is unique up to multiplication by an element of $\Gamma(U, \mathcal{O}_X^*)$.

Corollary (21.2.3). — *(i) Let \mathcal{J} be an invertible fractional ideal, and define \mathcal{J}' by $\mathcal{J}'|_U = \mathcal{O}_U f^{-1}$ for every U such that $\mathcal{J}|_U = \mathcal{O}_U f$. Then $\mathcal{J}^{-1} \cong \mathcal{J}'$.*

(ii) If $\mathcal{J}_1, \mathcal{J}_2$ are invertible fractional ideals, then $\mathcal{J}_1 \otimes \mathcal{J}_2 \cong \mathcal{J}_1 \mathcal{J}_2$.

(21.2.4). It follows that the set $\text{Idinv}(X)$ of fractional ideals forms a group under multiplication, with identity \mathcal{O}_X . The presheaf $U \mapsto \text{Idinv}(U)$ is a sheaf of abelian groups $\mathcal{I}dinv_X$.

(21.2.5). The map $f \mapsto \mathcal{J}(f) = \mathcal{O}_X f$ is a group homomorphism from $\Gamma(X, \mathcal{M}_X^*)$ to $\text{Idinv}(X)$. Considering this on open sets, we get a sheaf homomorphism

$$(21.2.5.1) \quad I_0: \mathcal{M}_X^* \rightarrow \mathcal{I}dinv_X$$

If $f \in \Gamma(X, \mathcal{O}_X^*)$ then $\mathcal{J}(f) = \mathcal{O}_X$, hence I_0 factors as

$$(21.2.5.2) \quad \mathcal{M}_X^* \rightarrow \mathcal{M}_X^*/\mathcal{O}_X^* = \mathcal{D}iv_X \xrightarrow{I} \mathcal{I}dinv_X.$$

In particular, we have a homomorphism $\mathcal{I}_U: \text{Div}(U) \rightarrow \text{Idinv}(U)$ such that $\mathcal{I}_U(\text{div}(f)) = \mathcal{O}_U f$, and for any divisor D , we have $f \in \Gamma(X, \mathcal{I}_X(D))$ iff $\text{div}(f) \geq D$.

Proposition (21.2.6). — *The homomorphism $I: \mathcal{D}iv_X \rightarrow \mathcal{I}dinv_X$ is bijective.*

(21.2.7). We identify $\mathcal{D}iv_X$ and $\mathcal{I}dinv_X$. For $D \in \text{Div}(X)$, we have $D \geq 0$ iff $\mathcal{I}_X(D) \subseteq \mathcal{O}_X$, so $\text{Div}^+(X)$ corresponds to the set of ideal sheaves $\mathcal{J} \subseteq \mathcal{O}_X$ such that \mathcal{J} is an invertible \mathcal{O}_X -module. [If X is integral, this amounts to \mathcal{J}_X being locally a principal ideal.] More generally, $D_1 \geq D_2$ iff $\mathcal{I}_X(D_1) \subseteq \mathcal{I}_X(D_2)$.

(21.2.8). We define $\mathcal{O}_X(D) = \mathcal{I}_X(D)^{-1}$.

(21.2.9). For any fractional ideal \mathcal{J} , the inclusion $\mathcal{J} \subseteq \mathcal{M}_X$ induces

$$(21.2.9.1) \quad \mathcal{M}_X(\mathcal{J}) = \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{M}_X \rightarrow \mathcal{M}_X \otimes_{\mathcal{O}_X} \mathcal{M}_X = \mathcal{M}_X.$$

In particular for any divisor D we have a canonical isomorphism

$$(21.2.9.2) \quad \mathcal{M}_X(\mathcal{O}_X(D)) \cong \mathcal{M}_X$$

and we denote by s_D the regular meromorphic section of $\mathcal{O}_X(D)$ corresponding to $1 \in \mathcal{M}_X$. Then

$$(21.2.9.3) \quad \text{div}(s_D) = D,$$

and we have

$$(21.2.9.4) \quad s_0 = 1, \quad s_{D+D'} = s_D \otimes s_{D'}, \quad s_{nD} = s_D^{\otimes n} \quad (n \in \mathbb{Z}).$$

(21.2.10). Let s, s' be regular meromorphic sections of invertible sheaves $\mathcal{L}, \mathcal{L}'$. Call $(\mathcal{L}, s), (\mathcal{L}', s')$ *equivalent* if there is an isomorphism $u: \mathcal{L} \rightarrow \mathcal{L}'$ carrying s to s' . Denote an equivalence class by $\text{cl}(\mathcal{L}, s)$ and let $D(X)$ denote the set of equivalence classes. Tensor product makes $D(X)$ an abelian group.

Proposition (21.2.11). — The groups $\text{Div}(X)$ and $D(X)$ are isomorphic, via

$$(21.2.11.1) \quad D \rightarrow \text{cl}(\mathcal{O}_X(D), s_D), \quad \text{cl}(\mathcal{L}, s) \rightarrow \text{div}(s).$$

(21.2.12). If $D \geq 0$ is an effective divisor on a prescheme X , then $\mathcal{I}_X(D)$ is an ideal sheaf in \mathcal{O}_X , defining a closed subscheme $Y(D) \subseteq X$. Locally we have $\mathcal{I}_X(D) = \mathcal{O}_U t$ for some regular section $t \in \Gamma(U, \mathcal{O})$, that is, the immersion $Y(D) \hookrightarrow X$ is *regular of codimension 1* at every point. We have $\text{Supp}(D) = Y(D)$. Conversely, every closed subscheme regular of codimension 1 is $Y(D)$ for an effective divisor D .

21.3. Linear equivalence of divisors.

(21.3.1). A divisor $D = \text{div}(f)$ of $f \in \Gamma(X, \mathcal{M}_X^*)$ is said to be *principal*. The principal divisors form a subgroup $\Gamma(X, \mathcal{M}_X^*)/\Gamma(X, \mathcal{O}_X^*)$ of $\text{Div}(X)$. If $D - D'$ is principal, D and D' are *linearly equivalent* [Liu, 7.1.17].

(21.3.2). Let $\text{Pic}(X)$ be the group of isomorphism classes of invertible \mathcal{O}_X -modules, under tensor product, writing $\text{cl}(\mathcal{L})$ for the isomorphism class of \mathcal{L} . Then

$$l_X: \text{cl}(\mathcal{L}, s) \mapsto \text{cl}(\mathcal{L})$$

is a homomorphism $\text{Div}(X) \cong D(X) \rightarrow \text{Pic}(X)$, and $l_X(D) = \text{cl}(\mathcal{O}_X(D))$.

Given $u: X' \rightarrow X$, we have a homomorphism $\text{Pic}(u): \text{Pic}(X) \rightarrow \text{Pic}(X')$ given by u^* .

Proposition (21.3.3). — (i) The kernel of $l_X: \text{Div}(X) \rightarrow \text{Pic}(X)$ consists of the principal divisors, i.e., $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ iff D is linearly equivalent to D' .

(ii) An invertible sheaf \mathcal{L} is of the form $l_X(D)$ iff there exists a regular meromorphic section of \mathcal{L} .

Proposition (21.3.4). — Let X be a prescheme satisfying either of:

(a) X is locally Noetherian and $\text{Ass}(X)$ is contained in some open affine $U \subseteq X$.

(b) X is reduced and its set of irreducible components is locally finite.

Then $l_X: \text{Div}(X) \rightarrow \text{Pic}(X)$ is surjective, i.e., $\text{Div}(X)/\text{Div.princ}(X) \cong \text{Pic}(X)$ [Liu, 7.1.19–20].

Corollary (21.3.5). — If X is a Noetherian prescheme on which there exists an ample sheaf (e.g., a quasi-projective scheme over Spec of a Noetherian ring), then $l_X: \text{Div}(X) \rightarrow \text{Pic}(X)$ is surjective.

[Proof: $\text{Ass}(X)$ is finite; apply (II, 4.5.4).]