Synopsis of material from EGA IV
§10: Jacobson Preschemes, 10.1-10.4

10.1 Very dense subsets of a topological space.

(10.1.1) A subset T of a topological space X is *quasi-constructible* if T is a finite union of locally closed subsets. T is *locally quasi-constructible* if every $x \in X$ has an open neighborhood V such that $T \cap V$ is quasi-constructible in V. The two notions are equivalent if X is quasi-compact. Let $\mathrm{Qc}(X)$, $\mathrm{Lqc}(X)$ denote the set of (locally) quasi-constructible subsets. Then $\mathrm{Qc}(X)$ and $\mathrm{Lqc}(X)$ are closed under finite intersections, unions, and complements, and preimages via continuous maps. Let $\mathrm{Op}(X)$ denote the set of open subsets of X, $\mathrm{Cl}(X)$ the set of closed subsets.

(10.1.2) *Proposition*: Let X_0 be a subspace of X. The following are equivalent.
(a) For every non-empty locally closed $Z \subseteq X$, $Z \cap X_0 \neq \emptyset$.
(a′) For every closed $Z \subseteq X$, $Z = \overline{Z \cap X}$.
(b) For every non-empty locally quasi-constructible $Z \subseteq X$, $Z \cap X_0 \neq \emptyset$.
(b′) For every locally quasi-constructible $Z \subseteq X$, $Z \subseteq \overline{Z \cap X_0}$.
(c) $U \mapsto U \cap X_0$ from $\mathrm{Op}(X)$ to $\mathrm{Op}(X_0)$ is injective (hence bijective).
(c′) $Z \mapsto Z \cap X_0$ from $\mathrm{Cl}(X)$ to $\mathrm{Cl}(X_0)$ is injective (hence bijective).
(c″) $Z \mapsto Z \cap X_0$ from $\mathrm{Qc}(X)$ to $\mathrm{Qc}(X_0)$ is injective (hence bijective).
(c‴) $Z \mapsto Z \cap X_0$ from $\mathrm{Lqc}(X)$ to $\mathrm{Lqc}(X_0)$ is injective (hence bijective).

(10.1.3) *Definition*: When the conditions in (10.1.2) hold, we say that X_0 is *very dense* in X.

(10.1.4) *Corollary*: If X_0 is very dense in X, and $U \subseteq X$ is open, then $U \cap X_0$ is very dense in U. Conversely, if $X = \bigcup_\alpha U_\alpha$ is an open covering such that $U_\alpha \cap X_0$ is very dense in U_α for each α, then X_0 is very dense in X.

Quasi-homeomorphisms.

(10.2.1) *Proposition*: Let $f\colon X_0 \to X$ be a continuous map. The following are equivalent.
(a) $U \mapsto f^{-1}(U)$ from $\mathrm{Op}(X)$ to $\mathrm{Op}(X_0)$ is bijective .
(a′) $Z \mapsto f^{-1}(Z)$ from $\mathrm{Cl}(X)$ to $\mathrm{Cl}(X_0)$ is bijective .
(b) The topology on X_0 is the inverse image of that on X, and $f(X_0)$ is very dense in X.
(c) The functor f^{-1} from sheaves on X to sheaves on X_0 is an equivalence of categories.

(10.2.2) *Definition*: A map f satisfying the conditions in (10.2.1) is a *quasi-homeomorphism*.

(10.2.3) *Corollary*: The composite of two quasi-homeomorphisms is a quasi-homeomorphism.

(10.2.4) *Corollary*: If $f\colon X \to Y$ is a quasi-homeomorphism, $Y' \subseteq Y$ is locally quasi-constructible, and $X' = f^{-1}(Y')$, then the restriction $f' = (f|X')\colon X' \to Y'$ is a quasi-homeomorphism.

(10.2.5) *Corollary*: Let $f\colon X \to Y$ be a continuous map, $Y = \bigcup_\alpha V_\alpha$ an open covering. If the restriction $f^{-1}(V_\alpha) \to V_\alpha$ of f is a quasi-homeomorphism for all α, then f is a quasi-homeomorphism.

(10.2.6) *Corollary*: Let $f\colon X \to Y$ be a quasi-homeomorphism, $Y' \subseteq Y$ locally quasi-constructible, $X' = f^{-1}(Y')$. Then Y' is quasi-compact (resp. Noetherian, retro-compact) iff X' is.

(10.2.7) *Proposition*: Let $f\colon X \to Y$ be a quasi-homeomorphism. Then the map $Z \mapsto f^{-1}(Z)$ from subsets of Y to subsets of X induces bijections between the open, closed, locally closed, quasi-constructible, locally quasi-constructible, constructible, and locally constructible subsets of X and Y.

(10.2.8) More generally, essentially all of the usual constructions of sheaf theory and sheaf cohomology are equivalent, whether carried out on X or on Y.

10.3 Jacobson spaces.

(10.3.1) *Definition*: A topological space X is *Jacobson* if the set of closed points X_0 of X is very dense in X; that is, if $X_0 \hookrightarrow X$ is a quasi-homeomorphism.

(10.3.2) *Proposition*: Let X be Jacobson, $Z \subseteq X$ locally quasi-constructible. Then the subspace Z is Jacobson, and a point $z \in Z$ is closed in Z iff it is closed in X.

(10.3.3) *Proposition*: Let $X = \bigcup_\alpha U_\alpha$ be an open covering. Then X is Jacobson iff every U_α is Jacobson.

10.4 Jacobson preschemes and Jacobson rings.

(10.4.1) *Definition*: A prescheme X is *Jacobson* if its underlying topological space is Jacobson. A ring A is *Jacobson* if $\operatorname{Spec}(A)$ is Jacobson.

A ring A is Jacobson in the sense of the definition iff every radical ideal of A is an intersection of maximal ideals; iff every prime ideal of A is an intersection of maximal ideals (the latter being the usual definition of a Jacobson ring).

(10.4.2) *Proposition*: Let $X = \bigcup_\alpha U_\alpha$ be an open affine covering of the prescheme X. Then X is Jacobson iff each ring $\mathcal{O}_X(U_\alpha)$ is Jacobson.

(10.4.3) Examples: a discrete space is Jacobson, hence an Artinian ring is Jacobson. A principal ideal domain with infinitely many maximal ideals (such as $\mathbb{Z}$) is Jacobson. A Noetherian local ring is Jacobson iff its maximal ideal is its only prime ideal; that is, iff it is Artinian. By (10.3.2), any sub-prescheme of a Jacobson scheme is Jacobson.

(10.4.4) *Proposition*: Let B be an integral domain. The following are equivalent.

(a) There exists $f \neq 0$ in B such that B_f is a field.

(b) The field of fractions of B is a finitely generated B-algebra.

(c) There exists a field K containing B, which is a finitely generated B-algebra.

(d) The generic point of $\operatorname{Spec}(B)$ is isolated (*i.e.*, the set consisting of only that point is open).

(d) $\Leftrightarrow$ (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) are easy. The significant point is (c) $\Rightarrow$ (b), which is a version of Hilbert's Nullstellensatz.

(10.4.5) *Proposition*: Given a ring A, the following are equivalent.

(a) A is Jacobson.

(b) For every non-maximal prime ideal $\mathfrak{p} \subseteq A$ and every $f \neq 0$ in $B = A/\mathfrak{p}$, B_f is not a field.

(b′) Every finitely generated A-algebra K which is a field, is finite over A (*i.e.*, finitely generated as an A-module; thus a finite algebraic extension of $A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal).

(10.4.6) *Corollary*: Every algebra B of finite type over a Jacobson ring A is Jacobson. Moreover, the preimage in A of any maximal ideal of B is maximal. In particular, any finitely generated algebra over $\mathbb{Z}$ or a field is Jacobson.

(10.4.7) *Corollary*: If X is a Jacobson prescheme and $f: Y \to X$ is a morphism locally of finite type, then Y is Jacobson, and f maps every closed point of X to a closed point of Y. Moreover, if $f(x) = y$, then $k(x)$ is a finite algebraic extension of $k(y)$.

(10.4.8) *Corollary*: If X is locally of finite type over an algebraically closed field k, then the k-rational points of X are very dense in X.

In fact, the k-rational points are the closed points, by (I, 6.4.2), and X is Jacobson.

(10.4.9-11) A number of questions in algebraic geometry can be reduced to the case of a finitely generated algebra over $\mathbb{Z}$ or a field, so the fact that such rings are Jacobson is particularly important. EGA gives two applications, of which the second is the following.

Proposition: Let X be an S-prescheme of finite type. Then any universally injective S-morphism $g: X \to X$ is bijective [where g is *universally injective* if it induces an injection $\underline{X}(K) \to \underline{X}(K)$ for every field K].

In fact, it is shown in (IV, 17.9.7) that g as above must be an isomorphism.