Synopsis of material from EGA I and II
II, §4: Projective bundles and ample sheaves, 4.5–6
I, §9: Supplement on quasi-coherent sheaves, 9.3–4

4.5. **Ample sheaves.**

(4.5.1). Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $S = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ is a positively graded subring of $\Gamma_*(\mathcal{L})$ (0, 5.4.6). Let $p\colon X \to \operatorname{Spec}(\mathbb{Z})$ be the structure morphism. We have a canonical graded $\mathcal{O}_X$-algebra homomorphism $\varepsilon\colon p^*(\widetilde{S}) \to \mathbf{S}(\mathcal{L}) = \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ by adjointness of $p_* = \Gamma(X, -)$ and p^*. Then (3.7.1) provides a canonical morphism $G(\varepsilon) \to \operatorname{Proj}(S)$.

When $\mathcal{L}$ is understood, define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for any $\mathcal{O}_X$-module $\mathcal{F}$.

Theorem (4.5.2). — *Let X be a quasi-compact scheme or a prescheme with Noetherian underlying space, and $\mathcal{L}$, S as above. The following are equivalent:*

(a) The sets X_f for homogeneous $f \in S_+$ form a base of the topology on X.

(a′) Those X_f which are affine cover X.

(b) The canonical morphism $G(\varepsilon) \to \operatorname{Proj}(S)$ is defined on all of X and is a dominant open immersion.

(b′) $G(\varepsilon) \to \operatorname{Proj}(S)$ is defined on all of X and is a homeomorphism of X onto a subspace of $\operatorname{Proj}(S)$.

(c) For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, let $\mathcal{F}_n$ be the submodule of $\mathcal{F}(n)$ generated by its global sections on X. Then $\mathcal{F}$ is the sum of its sub-$\mathcal{O}_X$-modules of the form $\mathcal{F}_n(-n)$, as n ranges over all positive integers.

(c′) Property (c) holds for quasi-coherent sheaves of ideals in $\mathcal{O}_X$.

Moreover, given homogeneous elements (f_α) in S_+ such that X_{f_α} is affine, the canonical morphism $X \to \operatorname{Proj}(S)$ restricts to an isomorphism $\bigcup_\alpha X_{f_\alpha} \cong \bigcup_\alpha D_+(f_\alpha) \subseteq \operatorname{Proj}(S)$.

[Proof: The preimage of $D_+(f)$ is X_f, and $G(\varepsilon)$ is the union of these. On any affine $U \subseteq X$ such that $\mathcal{L}|U \cong \mathcal{O}_U$ is trivial we have $X_f \cap U \cong U_{f'}$ for a section f' of $\mathcal{O}_U$ corresponding to f. So (b) ⇒ (b′) ⇒ (a) ⇒ (a′). By (I, 9.3.1–2) and (3.8.2), (a′) implies the "moreover," which together with (a′) implies (b). (I, 9.3.1) gives (a) ⇒ (c), clearly (c) ⇒ (c′), and (c) ⇒ (a) by taking for any open $U \subseteq X$ an ideal $\mathcal{J}$ such that $V(\mathcal{J})$ is the complement of U.]

Condition (b) implies that X is a *scheme*.

The proof also shows that those X_f which are affine form a base of the topology.

Definition (4.5.3). — An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is called *ample* if X is a quasi-compact scheme and the conditions in (4.5.2) hold.

By (a), if $\mathcal{L}$ is ample, then so is $\mathcal{L}|U$ for any quasi-compact open subset $U \subseteq X$.

Corollary (4.5.4). — *If $\mathcal{L}$ is ample, $Z \subseteq X$ is a finite subset, and U is a neighborhood of Z, there exists n and $f \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_f is an affine neighborhood of Z contained in U.*

[This uses a lemma from commutative algebra, that if $\mathfrak{p}_i$ are finitely many homogeneous prime ideals, not containing an ideal $I \subseteq S$, then there is a homogeneous element of I not contained in the union of the ideals $\mathfrak{p}_i$.]

Proposition (4.5.5). — *Let X be a quasi-compact scheme or a prescheme with Noetherian underlying space. The conditions in (4.5.2) are also equivalent to the following:*

(d) For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists n_0 such that $\mathcal{F}(n)$ is generated by its global sections for all $n \geq n_0$.

(d′) Every such $\mathcal{F}$ is isomorphic to a quotient of an $\mathcal{O}_X$-module of the form $\mathcal{L}^{\otimes(-n)} \otimes \mathcal{O}_X^k$.

(d″) Property (d′) holds for quasi-coherent ideal sheaves of finite type in $\mathcal{O}_X$.

[(c′) ⇒ (d) ⇒ (d′) ⇒ (d″) are straightforward. (d″) ⇒ (a) uses (9.4.9)]

Proposition (4.5.6). — *Let X be a quasi-compact scheme, $\mathcal{L}$ an inverible $\mathcal{O}_X$-module.*

(i) For $n > 0$, $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is ample.

(ii) Let $\mathcal{L}'$ be invertible and assume that for every $x \in X$ there exists $n > 0$ and $s \in \Gamma(X, \mathcal{L}'^{\otimes n})$ such that $s(x) \neq 0$. Then $\mathcal{L}$ ample implies $\mathcal{L} \otimes \mathcal{L}'$ ample.

Corollary (4.5.7). — *The tensor product of ample $\mathcal{O}_X$-modules is ample.*

Corollary (4.5.8). — *If $\mathcal{L}$ is ample, $\mathcal{L}'$ invertible, there exists $n_0 > 0$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is ample for all $n \geq n_0$.*

Remark (4.5.9). — In the [Picard group] $P \cong H^1(X, \mathcal{O}_X^*)$ of invertible sheaves on X, the ample sheaves form a subset P^+ such that

$$P_+ + P_+ \subseteq P_+, \quad P_+ - P_+ = P.$$

Hence P is a quasi-ordered abelian group with $P_+ \cup \{0\}$ its *positive* cone.

Proposition (4.5.10). — *Let Y be affine, $q\colon X \to Y$ quasi-compact and separated, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(i) If $\mathcal{L}$ is very ample for q, then $\mathcal{L}$ is ample.

(ii) Suppose q is of finite type*. Then $\mathcal{L}$ is ample iff the following equivalent conditions hold:*

(e) There exists $n_0 > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample for all $n \geq n_0$.

(e′) $\mathcal{L}^{\otimes n}$ is very ample for some $n > 0$.

(4.5.10.1). *Proof of Lemma (4.4.10.1).* — Let $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{K}^{\otimes n}$. For large n, we want to find a quasi-coherent subsheaf $\mathcal{F} \subseteq g_*(\mathcal{E}(n))$ of finite type such that the canonical map $g^*(\mathcal{F}) \to \mathcal{E}(n)$ is surjective. By quasi-compactness and (9.4.7), we can reduce to the case that Z is affine. Then (4.5.10, (i)) and (4.5.5, (d)) give the result.

Corollary (4.5.11). — *If Y is affine, $q\colon X \to Y$ separated and of finite type, $\mathcal{L}$ ample, $\mathcal{L}'$ invertible, there exists n_0 such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is very ample for q, for all $n \geq n_0$.*

Remark (4.5.12). — It is not known whether $\mathcal{L}^{\otimes n}$ very ample implies the same for $\mathcal{L}^{\otimes(n+1)}$.

Proposition (4.5.13). — *Let X be quasi-compact, $Z \subseteq X$ a closed sub-prescheme defined by a* nilpotent *sheaf of ideals, $j\colon Z \hookrightarrow X$ the inclusion. Then $\mathcal{L}$ is ample iff $\mathcal{L}' = j^*(\mathcal{L})$ is ample.*

[The proof relies on the following lemma, which in turn is proved using sheaf cohomology.]

Lemma (4.5.13.1). — *In (4.5.13), suppose further that $\mathcal{J}^2 = 0$, and let $g \in \Gamma(Z, \mathcal{L}'^{\otimes n})$ be such that Z_g is affine. Then there exists $m > 0$ such that $g^{\otimes m} = j^*(f)$ for a global section $f \in \Gamma(X, \mathcal{L}^{\otimes mn})$.*

Corollary (4.5.14). — *Let X be a Noetherian scheme, $j\colon X_{\mathrm{red}} \to X$ the inclusion. Then $\mathcal{L}$ is ample if and only if $j^*\mathcal{L}$ is ample.*

4.6. **Relatively ample sheaves.**

Definition (4.6.1). — Let $f\colon X \to Y$ be a quasi-compact morphism, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample relative to f*, or *f-ample*, or *ample relative to Y* (when f is understood) if there exists an open affine cover (U_α) of Y such that for every α, $\mathcal{L}|f^{-1}(U_\alpha)$ is ample.

Note that the existence of a relatively ample sheaf entails that f must be separated (4.5.3).

Proposition (4.6.2). — *Let $f\colon X \to Y$ be quasi-compact. If $\mathcal{L}$ is very ample for f, then $\mathcal{L}$ is ample relative to f.*

Proposition (4.6.3). — *Let $f\colon X \to Y$ be quasi-compact, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and put $\mathcal{S} = \bigoplus_{n\geq 0} f_*(\mathcal{L}^{\otimes n})$, a graded $\mathcal{O}_Y$-algebra. The following are equivalent:*

(a) $\mathcal{L}$ is f-ample.

(b) $\mathcal{S}$ is quasi-coherent and the canonical homomorphism $\sigma\colon f^*(\mathcal{S}) \to \mathbf{S}(\mathcal{L})$ (0, 4.4.3) induces an everywhere-defined, dominant open immersion $r_{\mathcal{L},\sigma}\colon X \hookrightarrow P = \mathrm{Proj}(\mathcal{S})$.

(b′) f is separated, and the morphism $r_{\mathcal{L},\sigma}$ is everywhere defined and is a homeomorphism of X onto a subspace of $\mathrm{Proj}(\mathcal{S})$.

Moreover, when these conditions hold, the canonical homomorphism $r^*_{\mathcal{L},\sigma}(\mathcal{O}_P(n)) \to \mathcal{L}^{\otimes n}$ (3.7.9.1) is an isomorphism. Furthermore, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, if we put $\mathcal{M} = \bigoplus_{n\geq 0} f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$, then $r^*_{\mathcal{L},\sigma}(\widetilde{\mathcal{M}}) \to \mathcal{F}$ (3.7.9.2) is an isomorphism.

Corollary (4.6.4). — *Let (U_α) be an open affine covering of Y. Then $\mathcal{L}$ is ample relative to f if and only if $\mathcal{L}|f^{-1}(U_\alpha)$ is ample relative to U_α, for all α.*

Corollary (4.6.5). — *Let $\mathcal{K}$ be an invertible $\mathcal{O}_Y$-module. Then $\mathcal{L}$ is f-ample iff $\mathcal{L} \otimes f^*(\mathcal{K})$ is.*

Corollary (4.6.6). — *Suppose Y affine. Then $\mathcal{L}$ is Y-ample iff it is ample.*

Corollary (4.6.7). — *Let $f\colon X \to Y$ be a quasi-compact morphism. Suppose there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and a morphism $g\colon X \to P = \mathrm{Proj}(\mathcal{E})$ which is a homeomorphism of X onto a subspace of P. Then $\mathcal{L} = g^*(\mathcal{O}_P(1))$ is f-ample.*

Proposition (4.6.8). — *Let X be a quasi-compact scheme or a prescheme with Noetherian underlying space, $f\colon X \to Y$ a quasi-compact, separated morphism. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is f-ample if and only if the following equivalent conditions hold:*

(c) For every $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists $n_0 > 0$ such that the canonical homomorphism $\sigma\colon f^*(f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is surjective for all $n \geq n_0$.

(c′) Property (c) holds for all $\mathcal{F} = \mathcal{J} \subseteq \mathcal{O}_X$ a quasi-coherent ideal sheaf of finite type.

Proposition (4.6.9). — *Let $f\colon X \to Y$ be a quasi-compact morphism, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(i) Let $n > 0$. Then $\mathcal{L}$ is f-ample iff $\mathcal{L}^{\otimes n}$ is.

(ii) Let $\mathcal{L}'$ be an invertible $\mathcal{O}_X$-module such that $\sigma\colon f^*(f_*(\mathcal{L}'^{\otimes n})) \to \mathcal{L}'^{\otimes n}$ for some $n > 0$. Then if $\mathcal{L}$ is f-ample, so is $\mathcal{L} \otimes \mathcal{L}'$.

Corollary (4.6.10). — *The tensor product of f-ample $\mathcal{O}_X$-module is f-ample.*

Proposition (4.6.11). — *Let Y be quasi-compact, $f\colon X \to Y$ a morphism of finite type, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample iff the following equivalent conditions hold:*

(d) There exists $n_0 > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample for f, for all $n \geq n_0$.

(d′) There exists $n > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample for f.

Corollary (4.6.12). — *Let Y be quasi-compact, $f\colon X \to Y$ of finite type, $\mathcal{L}$, $\mathcal{L}'$ invertible $\mathcal{O}_X$-modules. If $\mathcal{L}$ is f-ample, there exists n_0 such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is very ample for f, for all $n \geq n_0$.*

Proposition (4.6.13). — *(i) Every invertible $\mathcal{O}_Y$-module $\mathcal{L}$ is ample relative to the identity map $1_Y\colon Y \to Y$.*

(i′) Let $f\colon X \to Y$ be quasi-compact, $j\colon X' \to X$ a quasi-compact morphism which is a homeomorphism of X' onto a subspace of X. If $\mathcal{L}$ is f-ample, then $j^*\mathcal{L}$ is ample relative to $f \circ j$.

(ii) Let Z be quasi-compact, $f\colon X \to Y$, $g\colon Y \to Z$ quasi-compact morphisms, $\mathcal{L}$ f-ample, $\mathcal{K}$ g-ample. Then there exists $n_0 > 0$ such that $\mathcal{L} \otimes f^*(\mathcal{K}^{\otimes n})$ is ample relative to $g \circ f$, for all $n \geq n_0$.

(iii) Let $f\colon X \to Y$ be quasi-compact $g\colon Y' \to Y$ any morphism. If $\mathcal{L}$ is f-ample, then $\mathcal{L} \otimes_Y \mathcal{O}_{Y'}$ is ample relative to $f_{(Y')}$.

(iv) Let $f_i\colon X_i \to Y_i$ $(i = 1, 2)$ be quasi-compact S-morphisms. If $\mathcal{L}_i$ is ample relative to f_i, then $\mathcal{L}_1 \otimes_S \mathcal{L}_2$ is ample relative to $f_1 \times_S f_2$.

(v) Let $f\colon X \to Y$, $g\colon Y \to Z$, be such that $g \circ f$ is quasi-compact. Assume that g is separated, or that X has locally Noetherian underlying space. If $\mathcal{L}$ is ample relative to $g \circ f$, then $\mathcal{L}$ is f-ample.

(vi) Let $f\colon X \to Y$ be quasi-compact, $j\colon X_{\mathrm{red}} \hookrightarrow X$ the inclusion. If $\mathcal{L}$ is f-ample, then $j^*\mathcal{L}$ is ample relative to f_{red}.

[Assertions (i), (i′), (iii) and (iv) imply the rest; (i) is trivial from (4.4.10, (i)) and (4.6.2). The others are proved using the following lemma.]

Lemma (4.6.13.1). — *(i) Let $u\colon Z \to S$ be a morphism $\mathcal{L}$ an invertible $\mathcal{O}_S$-module, $\mathcal{L}' = u^*(\mathcal{L})$, $s \in \Gamma(S, \mathcal{L})$, $s' = u^*(s)$. Then $Z_{s'} = u^{-1}(S_s)$.*

(ii) Let Z, Z' be S-preschemes, $T = Z \times_S Z'$, p, p' the projections, $\mathcal{L}$ (resp. $\mathcal{L}$) and invertible $\mathcal{O}_Z$-module (resp. $\mathcal{O}_{Z'}$-module), $t \in \Gamma(Z, \mathcal{L})$, $t' \in \Gamma(Z', \mathcal{L}')$, $s = p^*(t)$, $s' = p'^*(t')$. Then $T_{s \otimes s'} = Z_t \times_S Z'_{t'}$.

Remark (4.6.14). — In (ii) it need not be the case that $\mathcal{L} \otimes f^*(\mathcal{K})$ is ample relative to $g \circ f$. Were this so, one could take $\mathcal{L}' = \mathcal{L} \otimes f^*(\mathcal{K}^{-1})$ in the place of $\mathcal{L}$ and conclude that $\mathcal{L}$

is ample relative to $g \circ f$, for any invertible $\mathcal{O}_X$-module $\mathcal{L}$, which is clearly false (suppose g were the identity!).

Proposition (4.6.15). — *Let $f: X \to Y$ be quasi-compact, $\mathcal{J} \subseteq \mathcal{O}_X$ a locally nilpotent quasi-coherent ideal sheaf, $j: Z = V(\mathcal{J}) \hookrightarrow X$ the inclusion of the closed subscheme defined by $\mathcal{J}$. Then $\mathcal{L}$ is ample for f if and only if $j^*(\mathcal{L})$ is ample for $f \circ j$.*

Corollary (4.6.16). — *Let X be locally Noetherian, $f: X \to Y$ quasi-compact, $j: X_{\mathrm{red}} \hookrightarrow X$ the inclusion. Then $\mathcal{L}$ is ample for f if and only if $j^*(\mathcal{L})$ is ample for f_{red}.*

Proposition (4.6.17). — *With the notation and hypotheses of (4.4.11), $\mathcal{L}''$ is ample relative to f'' iff $\mathcal{L}$ is ample relative to f and $\mathcal{L}'$ is ample relative to f'.*

Proposition (4.6.18). — *Let Y be quasi-compact, $\mathcal{S}$ a graded quasi-coherent $\mathcal{O}_Y$-algebra of finite type, $X = \mathrm{Proj}(\mathcal{S})$, $f: X \to Y$ the structure morphism. Then f is of finite type, and $\mathcal{O}_X(d)$ is invertible and f-ample for some $d > 0$.*

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9.3. **Extending sections of quasi-coherent sheaves.**

Theorem (9.3.1). — *Let X be a quasi-compact scheme or a prescheme with Noetherian underlying space. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, $f \in \Gamma(X, \mathcal{L})$, $X_f = \{x \in X : f(x) \neq 0\}$ (0, 5.5.1), $\mathcal{F}$ a quasi-coherent sheaf.*

(i) If $s \in \Gamma(X, \mathcal{F})$ satisfies $s|X_f = 0$, then $s \otimes f^{\otimes n} = 0$ for some $n > 0$.

(ii) For every $s \in \Gamma(X_f, \mathcal{F})$, there exists $n > 0$ such that $s \otimes f^{\otimes n}$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

Corollary (9.3.2). — *Let $A = \Gamma_*(\mathcal{L})$ (which is a graded ring), $M = \Gamma_*(\mathcal{L}, \mathcal{F})$ (which is a graded A-module (0, 5.4.6)). For any $f \in A_n$, there is a canonical isomorphism $\Gamma(X_f, \mathcal{F}) \cong (M_f)_0$.*

Corollary (9.3.3). — *Let $\mathcal{L} = \mathcal{O}_X$, so $f \in \Gamma(X, \mathcal{O}_X)$. If $A = \Gamma(X, \mathcal{O}_X)$, $M = \Gamma(X, \mathcal{F})$, then $\Gamma(X_f, M) \cong M_f$.*

Proposition (9.3.4). — *If X is Noetherian, $\mathcal{F}$ coherent, $\mathcal{J} \subseteq \mathcal{O}_X$ a coherent sheaf of ideals such that the support of $\mathcal{F}$ is contained in $V(\mathcal{J})$, then $\mathcal{J}^n\mathcal{F} = 0$ for some $n > 0$.*

Corollary (9.3.5). — *In (9.3.4), there exists a closed subscheme $j: Y \hookrightarrow X$ with underlying space equal to $V(\mathcal{J})$, such that $\mathcal{F} = j_*(j^*(\mathcal{F}))$.*

9.4. **Extending quasi-coherent sheaves.**

(9.4.1). Let $\mathcal{F}$ be a sheaf of sets, groups, or rings on X, $\psi: U \hookrightarrow X$ an open set, $\mathcal{G}$ a subsheaf of $\mathcal{F}|U$. Then $\psi_*(\mathcal{G})$ is a subsheaf of $\psi_*(\psi^{-1}(\mathcal{F}))$. With $\rho: \mathcal{F} \to \psi_*(\psi^{-1}(\mathcal{F}))$ the canonical map, let $\overline{\mathcal{G}} = \rho^{-1}(\psi_*(\mathcal{G}))$. The sections of $\overline{\mathcal{G}}$ are the sections of $\mathcal{F}$ whose restriction to U belong to $\mathcal{G}$, thus $\overline{\mathcal{G}}$ is the maximal subsheaf of $\mathcal{F}$ whose restriction to U is equal to $\mathcal{G}$.

Proposition (9.4.2). — *Let $j\colon U \to X$ be the inclusion of an open subscheme of a prescheme X, and assume that j is a quasi-compact morphism (this always holds if the underlying space of X is locally Noetherian).*

(i) If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_U$-module, then $j_*(\mathcal{G})$ is quasi-coherent and $j_*(\mathcal{G})|U = j^{-1}(j_*(\mathcal{G})) = \mathcal{G}$.

(ii) If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and $\mathcal{G}$ is a quasi-coherent subsheaf of $\mathcal{F}|U$, then $\overline{\mathcal{G}}$ is quasi-coherent.

Corollary (9.4.3). — *Let $U \subseteq X$ be quasi-compact and suppose the inclusion $j\colon U \hookrightarrow X$ is quasi-compact. Suppose further that every quasi-coherent $\mathcal{O}_X$-module is the direct limit of its quasi-coherent submodules of finite type (for instance, if X is affine). If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{G} \subseteq \mathcal{F}|U$ is a quasi-coherent submodule of finite type, then there exists a quasi-coherent submodule $\mathcal{G}' \subseteq \mathcal{F}$ of finite type such that $\mathcal{G} = \mathcal{G}'|U$.*

Remark (9.4.4). — Suppose that for every affine open $U \subseteq X$, the inclusion $U \hookrightarrow X$ is quasi-compact. If the conclusion of (9.4.3) holds for every affine $U \subseteq X$ and quasi-coherent submodule $\mathcal{G} \subseteq \mathcal{F}|U$ of finite type, then $\mathcal{F}$ is necessarily the direct limit of its quasi-coherent subsheaves of finite type.

Corollary (9.4.5). — *Under the hypotheses of (9.4.3), every quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$ of finite type is the restriction of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}'$ of finite type.*

Lemma (9.4.6). — *Let $(V_\lambda)_{\lambda\in L}$ be a covering of X by open affines, where the index set L is well-ordered. Let $U \subseteq X$ be an open set. For every $\lambda \in L$, put $W_\lambda = \bigcup_{\mu<\lambda} V_\mu$. Suppose (1) $V_\lambda \cap W_\lambda$ is quasi-compact for all λ, and (2) $U \hookrightarrow X$ is a quasi-compact morphism. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and quasi-coherent submodule $\mathcal{G} \subseteq \mathcal{O}_U$ of finite type, there exists a quasi-coherent submodule $\mathcal{G}' \subseteq \mathcal{F}$ of finite type such that $\mathcal{G} = \mathcal{G}'|U$.*

Theorem (9.4.7). — *Let $U \subseteq X$ be an open subset of a prescheme X, and suppose either of the following holds:*

(a) The underlying space of X is locally Noetherian.

(b) X is a quasi-compact scheme and U is quasi-compact.

Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and quasi-coherent submodule $\mathcal{G} \subseteq \mathcal{O}_U$ of finite type, there exists a quasi-coherent submodule $\mathcal{G}' \subseteq \mathcal{F}$ of finite type such that $\mathcal{G} = \mathcal{G}'|U$.

Corollary (9.4.8). — *Under the hypotheses of (9.4.7), for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$ of finite type, there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite type such that $\mathcal{G} = \mathcal{G}'|U$.*

Corollary (9.4.9). — *Let X be a quasi-compact scheme or a prescheme with locally Noetherian underlying space. Then every quasi-coherent $\mathcal{O}_X$-module is the direct limit of its quasi-coherent submodules of finite type.*

Corollary (9.4.10). — *Under the hypotheses of (9.4.9), if a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ has the property that every quasi-coherent submodule of $\mathcal{F}$ of finite type is generated by its global sections, then $\mathcal{F}$ is generated by its global sections.*