Synopsis of material from EGA II
§4: Projective bundles and ample sheaves, 4.1–4

4.1. **Definition of projective bundles.**

Definition (4.1.1). — Let $\mathbf{S}(\mathcal{E})$ be the symmetric algebra of a quasi-coherent $\mathcal{O}_Y$-module. The *projective bundle over* Y *defined by* $\mathcal{E}$ is the Y-scheme $\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathbf{S}(\mathcal{E}))$. The twisting sheaf $\mathcal{O}(1)$ on $\mathbf{P}(\mathcal{E})$ is its *fundamental sheaf*.

If Y is affine, $\mathcal{E} = \widetilde{E}$, we also write $\mathbf{P}(E)$. If $\mathcal{E} = \mathcal{O}_Y^n$, we put $\mathbf{P}_Y^{n-1} = \mathbf{P}(\mathcal{E})$, also denoted $\mathbf{P}_A^{n-1}$ if $Y = \mathrm{Spec}(A)$.

(4.1.2). A surjective homomorphism $\mathcal{E} \to \mathcal{F}$ induces a closed immersion $j\colon Q = \mathbf{P}(\mathcal{F}) \hookrightarrow \mathbf{P}(\mathcal{E}) = P$, such that $j^*\mathcal{O}_P(n) = \mathcal{O}_Q(n)$ [(3.6.2–3)].

(4.1.3). Given a morphism $\psi\colon Y' \to Y$, we have $P' = \mathbf{P}(\psi^*\mathcal{E}) = \mathbf{P}(\mathcal{E}) \otimes_Y Y'$, and $\mathcal{O}_{P'}(n) = \mathcal{O}_P(n) \otimes_Y \mathcal{O}_{Y'}$ [(3.5.3–4)].

Proposition (4.1.4). — *If $\mathcal{L}$ is invertible, we have an isomorphism $i\colon P = \mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{E} \otimes \mathcal{L}) = Q$, and $i^*\mathcal{O}_Q(n) = \mathcal{O}_P(n) \otimes_Y \mathcal{L}^{\otimes n}$ [(3.1.8, (iii)), (3.2.10)].*

(4.1.5). Let $p\colon P = \mathbf{P}(\mathcal{E}) \to Y$ be the structure morphism. Since $\mathcal{E} = \mathbf{S}(\mathcal{E})_1$, we have canonical homomorphisms $\alpha_1\colon \mathcal{E} \to p_*\mathcal{O}_P(1)$ (3.3.2) and [by (0, 4.4.3)]

$$\alpha_1^\sharp\colon p^*(\mathcal{E}) \to \mathcal{O}_P(1). \tag{4.1.5.1}$$

Proposition (4.1.6). — *The canonical homomorphism (4.1.5.1) is surjective [(3.2.4)].*

4.2. **Morphisms from a prescheme to a projective bundle.**

(4.2.1). Keep the notation of (4.1.5). Let $q\colon X \to Y$ be a Y-prescheme, $r\colon X \to P$ a Y-morphism. Then $\mathcal{L}_r = r^*\mathcal{O}_P(1)$ is an invertible sheaf on X, and we deduce from (4.1.5.1) a canonical surjection

$$\phi_r\colon q^*(\mathcal{E}) \to \mathcal{L}_r. \tag{4.2.1.1}$$

Suppose $Y = \mathrm{Spec}(A)$, $\mathcal{E} = \widetilde{E}$, $f \in E$, so $r^{-1}(D_+(f)) = X_{\phi_r^\flat(f)}$ by (2.6.3), $U = \mathrm{Spec}(B) \subseteq X_{\phi_r^\flat(f)}$. On U, r corresponds to a ring homomorphism $S_{(f)} \to B$, where $S = \mathbf{S}(E)$. We have $q^*(\mathcal{E})|U = (E \otimes_A B)\widetilde{}$ and $\mathcal{L}_r|U = \widetilde{L}_r$, where $L_r = S(1)_{(f)} \otimes_{S_{(f)}} B$. Then ϕ_r corresponds to $E \otimes_A B \to L_r$ given by $x \otimes 1 \mapsto (f/1) \otimes (x/f)$.

(4.2.2). Conversely, suppose given $q\colon X \to Y$, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, and a homomorphism $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$. Then we get an $\mathcal{O}_X$-algebra homomorphism $\psi\colon q^*(\mathbf{S}(\mathcal{E})) \to \mathbf{S}(\mathcal{L})$, inducing a Y-morphism $r_{\mathcal{L},\psi}\colon G(\psi) \to \mathbf{P}(\mathcal{E})$ as in (3.7.1). If ϕ is sujective, then so is ψ, and $r_{\mathcal{L},\psi}$ is defined on all of X.

Proposition (4.2.3). — *Given $q\colon X \to Y$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$, Y-morphisms $r\colon X \to \mathbf{P}(\mathcal{E})$ correspond bijectively to equivalence classes of surjective $\mathcal{O}_X$-module homomorphisms $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$ with $\mathcal{L}$ invertible, where $(\mathcal{L}, \phi)$, $(\mathcal{L}, \phi')$ are equivalent if there is an isomoprhism $\tau\colon \mathcal{L} \to \mathcal{L}'$ such that $\phi' = \tau \circ \phi$.*

Theorem (4.2.4). — *The set of Y-sections of $\mathbf{P}(\mathcal{E})$ corresponds bijectively with the set of quasi-coherent subsheaves $\mathcal{F} \subseteq \mathcal{E}$ such that $\mathcal{E}/\mathcal{F}$ is invertible.* [Special case of (4.2.3) with $X = Y$.]

If $Y = \mathrm{Spec}(k)$ this identifies the k-points of $\mathbf{P}_k^{n-1}$ with the set of codimension-1 subspaces $F \subseteq k^n$.

Remark (4.2.5). — Proposition (4.2.3) says that the functor represented by $\mathbf{P}(\mathcal{E})$ on the category of Y-preschemes is naturally isomorphic to the functor that assigns to X the set of quasi-coherent subsheaves $\mathcal{F} \subseteq q^*(\mathcal{E})$ such that the quotient is invertible (pullback via $\psi\colon X' \to X$ makes this a functor in an obvious way). Later we'll see that "Grassmann schemes," for instance, can be defined similarly.

Corollary (4.2.6). — *Suppose that every invertible $\mathcal{O}_Y$-module is trivial. Let $A = \Gamma(Y, \mathcal{O}_Y)$, and $V = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y)$, regarded as an A-module. Let V^* be the subset of surjections in V, A^* the group of units in A. Then the set of Y-sections of $\mathbf{P}(\mathcal{E})$ is identified with V^*/A^*.*

The hypothesis holds for any local scheme Y (I, 2.4.8). For any extension K of $k(y)$, the set of K-points of the fiber $p^{-1}(y)$ of $\mathbf{P}(\mathcal{E})$ is identified (4.1.3.1) with the projective space of codimension-1 subspaces in the vector space $\mathcal{E}(y) \otimes_{k(y)} K$, where $\mathcal{E}(y) = \mathcal{E} \otimes_{\mathcal{O}_Y} k(y) = \mathcal{E}/\mathfrak{m}_y\mathcal{E}$.

If $Y = \mathrm{Spec}(A)$ and all invertible $\mathcal{O}_Y$-modules are trivial [e.g., if A is a UFD], then when $\mathcal{E} = \mathcal{O}_Y^n$, we have $V = A^n$ in (4.2.6), V^* consists of systems $(f_1, \ldots, f_n)$ which generate the unit ideal in A, and two such define the same Y-section of $\mathbf{P}_A^{n-1}$ if they differ by multipication by a unit of A.

Thus $\mathbf{P}(\mathcal{E})$ generalizes the classical concept of projective space.

(4.2.7). [Later we will see how to determine the group of invertible sheaves on $\mathbf{P}(\mathcal{E})$, which leads to a description of the sheaf of local automorphisms of $\mathbf{P}(\mathcal{E})$ over Y as a quotient of the sheaf of groups $\mathcal{A}ut(\mathcal{E})$.]

(4.2.8). Keep the notation of (4.2.1). If $u\colon X' \to X$ is a morphism, and $r\colon X \to P$ corresponds to $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$, then $r \circ u$ corresponds to $u^*(\phi)$.

(4.2.9). Suppose $v\colon \mathcal{E} \to \mathcal{F}$ is surjective, and let $j\colon \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E})$ be the corresponding closed immersion (4.1.2). If $r\colon X \to \mathbf{P}(\mathcal{F})$ corresponds to $\phi\colon q^*(\mathcal{F}) \to \mathcal{L}$, then $j \circ r$ corresponds to $\phi \circ q^*(v)$.

(4.2.10). Given $\psi\colon Y' \to Y$ and $r\colon X \to P$, the base extension $r_{(Y')}\colon X_{(Y')} \to P' = \mathbf{P}(\mathcal{E}')$, where $\mathcal{E}' = \psi^*(\mathcal{E})$, corresponds to $\phi_{(Y')} = \phi \otimes_{\mathcal{O}_Y} 1_{\mathcal{O}_{Y'}}$.

4.3. **The Segre morphism.**

(4.3.1). Let $\mathcal{E}$, $\mathcal{F}$ be quasi-coherent $\mathcal{O}_Y$-modules. Set $P_1 = \mathbf{P}(\mathcal{E})$, $P_2 = \mathbf{P}(\mathcal{F})$, with structure morphisms $p_i\colon P_i \to Y$. Let $Q = P_1 \times_Y P_2$, with projections $q_i\colon Q \to P_i$. Let $\mathcal{L} = \mathcal{O}_{P_1}(1) \otimes_Y \mathcal{O}_{P_2}(1) = q_1^*(\mathcal{O}_{P_1}(1)) \otimes_{\mathcal{O}_Q} q_2^*(\mathcal{O}_{P_2}(1))$, an invertible $\mathcal{O}_Q$-module. Then $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structure morphism $Q \to Y$, and the canonical surjections $p_i^*(\mathcal{E}) \to \mathcal{O}_{P_i}(1)$ give rise to a surjection

$$s\colon r^*(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) \to \mathcal{L}. \tag{4.3.1.1}$$

By (4.2.2) this induces a morphism, the *Segre morphism*

(4.3.1.2) $$\zeta\colon \mathbf{P}(\mathcal{E}) \times_Y \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}).$$

Set $P = \mathbf{P}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F})$. Making things explicit for Y affine, $\mathcal{E} = \widetilde{E}$, $\mathcal{F} = \widetilde{F}$, one shows that

$$\zeta^{-1}(P_{x\otimes y}) = (P_1)_x \times_Y (P_2)_y,$$

which comes down to the following easy lemma.

Lemma (4.3.2). — *Given A-algebras B, B', and elements $t \in B$, $t \in B'$, one has $D(t \otimes t') = D(t) \times_Y D(t')$ in* $\operatorname{Spec}(B) \times_A \operatorname{Spec}(B')$.

Proposition (4.3.3). — *The Segre morphism is a closed immersion.*

(4.3.4). The Segre morphism is functorial with respect to closed immersions $\mathbf{P}(\mathcal{E}') \hookrightarrow \mathbf{P}(\mathcal{E})$, $\mathbf{P}(\mathcal{F}') \hookrightarrow \mathbf{P}(\mathcal{F})$ induced by surjections $\mathcal{E} \to \mathcal{E}'$, $\mathcal{F} \to \mathcal{F}'$.

(4.3.5). The Segre morphism commutes with base extension by $\psi\colon Y' \to Y$.

Remark (4.3.6). — There is also a canonical closed immersion of the disjoint union of $\mathbf{P}(\mathcal{E})$, $\mathbf{P}(\mathcal{F})$ into $\mathbf{P}(\mathcal{E} \oplus \mathcal{F})$.

4.4. **Immersions into projective bundles. Very ample sheaves.**

Proposition (4.4.1). — *Let Y be a quasi-compact scheme or a prescheme with Noetherian underlying space, $q\colon X \to Y$ a morphism of finite type, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(i) Let $\mathcal{S}$ be a graded quasi-coherent $\mathcal{O}_Y$-algebra, and $\psi\colon q^*(\mathcal{S}) \to \mathbf{S}(\mathcal{L})$ a graded $\mathcal{O}_X$-algebra homomorphism. Then $r_{\mathcal{L},\psi}$ is an everywhere defined immersion iff there exist n and a quasi-coherent submodule $\mathcal{E}$ of finite type *in $\mathcal{S}_n$, such that the induced homomorphism $q^*(\mathcal{E}) \to \mathcal{L}^{\otimes n}$ is surjective and the corresponding morphism $r\colon X \to \mathbf{P}(\mathcal{E})$ is an immersion.*

(ii) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-module and $\phi\colon q^*(\mathcal{F}) \to \mathcal{L}$ a surjection. Then $r_{\mathcal{L},\phi}$ is an immersion if and only if there is a quasi-coherent sub-sheaf $\mathcal{E} \subseteq \mathcal{F}$ of finite type *such that $\phi'\colon q^*(\mathcal{E}) \to \mathcal{L}$ is surjective and $r_{\mathcal{L},\phi'}$ is an immersion.*

[The proof uses (3.8.5).]

Definition (4.4.2). — Given $q\colon X \to Y$, an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *very ample* (for q) if there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and an immersion of Y-schemes $i\colon X \hookrightarrow P = \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} \cong i^*\mathcal{O}_P(1)$.

Equivalently, there exists a surjection $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$ such that $r_{\mathcal{L},\phi}$ is an immersion. Note that the existence of a very ample sheaf entails that q must be *separated* (3.1.3).

Corollary (4.4.3). — *If $\mathcal{L} \cong i^*\mathcal{O}_P(1)$ for an immersion $i\colon X \to P =$ $\operatorname{Proj}(\mathcal{S})$, *where $\mathcal{S}$ is a graded quasi-coherent $\mathcal{O}_Y$-algebra generated by $\mathcal{S}_1$, then $\mathcal{L}$ is very ample.*

Proposition (4.4.4). — *Suppose $q\colon X \to Y$ quasi-compact, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. The following are equivalent:*

(a) $\mathcal{L}$ is very ample for q;

(b) $q_*(\mathcal{L})$ is quasi-coherent, the canonical homomorphism $\sigma\colon q^*(q_*(\mathcal{L})) \to \mathcal{L}$ is surjective, and $r_{\mathcal{L},\sigma}\colon X \to \mathbf{P}(q_*(\mathcal{L}))$ is an immersion.

Recall that since q is quasi-compact, $q_*(\mathcal{L})$ is quasi-coherent if q is separated.

Corollary (4.4.5). — *Suppose q quasi-compact. If there exists an open covering (U_α) of Y such that $\mathcal{L}|q^{-1}(U_\alpha)$ is very ample relative to U_α, for all α, then $\mathcal{L}$ is very ample.*

Proposition (4.4.6). — *Let Y be a quasi-compact scheme or a prescheme with Noetherian underlying space, $q\colon X \to Y$ a morphism of finite type, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then the conditions of (4.4.4) are also equivalent to:*

(a′) There exists an $\mathcal{O}_Y$-module $\mathcal{E}$ of finite type *and a surjection $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$ such that $r_{\mathcal{L},\phi}$ is an immersion.*

(b′) There exists a quasi-coherent sub-$\mathcal{O}_Y$-module $\mathcal{E} \subseteq q_*(\mathcal{L})$ of finite type *with the property in (a′).*

Corollary (4.4.7). — *Suppose Y is a quasi-compact scheme or a Noetherian prescheme. If $\mathcal{L}$ is very ample for q, then there exists a graded quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{S}$, such that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$, and an* open, dominant *Y-immersion $i\colon X \to P = \mathrm{Proj}(\mathcal{S})$ such that $\mathcal{L} \cong i^*\mathcal{O}_P(1)$.*

Proposition (4.4.8). — *Let $\mathcal{L}$ be very ample for $q\colon X \to Y$, $\mathcal{L}'$ any invertible $\mathcal{O}_X$-module such that there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and a surjection $q^*(\mathcal{E}) \to \mathcal{L}'$. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is very ample.*

Corollary (4.4.9). — *Let $q\colon X \to Y$ be a morphism.*

(i) Given an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and invertible $\mathcal{O}_Y$-module $\mathcal{M}$, $\mathcal{L}$ is very ample if and only if $\mathcal{L} \otimes_{\mathcal{O}_X} q^*(\mathcal{M})$ is.

(ii) If $\mathcal{L}$ and $\mathcal{L}'$ are very ample, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$; in particular $\mathcal{L}^{\otimes n}$ is very ample for all $n > 0$.

Proposition (4.4.10). — *(i) Every invertible $\mathcal{O}_Y$-module $\mathcal{L}$ is very ample for the identity map $1_Y\colon Y \to Y$.*

(i′) Given $f\colon X \to Y$ and an immersion $j\colon X' \to X$, if $\mathcal{L}$ is very ample for f, then $j^*\mathcal{L}$ is very ample for $f \circ j$.

(ii) Let Z be quasi-compact, $f\colon X \to Y$ a morphism of finite type, $g\colon Y \to Z$ a quasi-compact morphism, $\mathcal{L}$ very ample for f, $\mathcal{K}$ very ample for g. Then there exists $n_0 > 0$ such that $\mathcal{L} \otimes f^*(\mathcal{K}^{\otimes n})$ is very ample for $g \circ f$, for all $n \geq n_0$.

(iii) Given $f\colon X \to Y$, $g\colon Y' \to Y$, if $\mathcal{L}$ is very ample for f, then $\mathcal{L} \otimes_Y \mathcal{O}_{Y'}$ is very ample for $f_{(Y')}$.

(iv) Given two S-morphisms $f_i\colon X_i \to Y_i$ ($i = 1, 2$), if $\mathcal{L}_i$ is very ample for f_i, then $\mathcal{L}_1 \otimes_S \mathcal{L}_2$ is very ample for $f_1 \times_S f_2$.

(v) Given $f\colon X \to Y$, $g\colon Y \to Z$, if $\mathcal{L}$ is very ample for $g \circ f$, then $\mathcal{L}$ is very ample for f.

(vi) If $\mathcal{L}$ is very ample for $f\colon X \to Y$, then $j^*\mathcal{L}$ is very ample for f_{red}, where $j\colon X_{\mathrm{red}} \hookrightarrow X$ is the canonical injection.

[The proof of (ii) uses the following lemma, proved in §4.5]

Lemma (4.4.10.1).— *Let Z be a quasi-compact scheme or a prescheme with Noetherian underlying space, $g\colon Y \to Z$ a quasi-compact morphism, $\mathcal{K}$ very ample for g, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_Y$-module of finite type. Then there exists m_0 such that for all $m \geq m_0$, $\mathcal{E}$ is isomorphic to a quotient of an $\mathcal{O}_Y$-module of the form $g^*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{K}^{\otimes -m}$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Z$-module of finite type (depending on m).*

[Then it is shown that if $f^*(\mathcal{E}) \to \mathcal{L}$ induces an immersion $X \to \mathbf{P}(\mathcal{E})$, and there is a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$ and a surjection $g^*(\mathcal{F}) \to \mathcal{E} \otimes \mathcal{K}^{\otimes m}$, then $\mathcal{L} \otimes \mathcal{K}^{\otimes(m+1)}$ is very ample for $X \to Z$.]

Proposition (4.4.11). — *Let $X'' = X \bigsqcup X'$ be a prescheme disjoint union, $f''\colon X'' \to Y$ a morphism restricting to morphisms $f\colon X \to Y$, $f'\colon X' \to Y$. Let $\mathcal{L}$, $\mathcal{L}'$ be invertible $\mathcal{O}_X$, $\mathcal{O}_{X'}$-modules, $\mathcal{L}''$ the invertible $\mathcal{O}_{X''}$-module restricting to $\mathcal{L}$, $\mathcal{L}'$. Then $\mathcal{L}''$ is very ample iff $\mathcal{L}$ and $\mathcal{L}'$ are very ample.*