

SYNOPSIS OF MATERIAL FROM EGA II

§3: HOMOGENEOUS PRIME SPECTRUM OF A SHEAF OF GRADED ALGEBRAS

3.1. Homogeneous spectrum of a graded quasi-coherent \mathcal{O}_Y -algebra.

(3.1.1). Let Y be a prescheme. A sheaf of graded \mathcal{O}_Y -algebras $\mathcal{S} = \bigoplus_n \mathcal{S}_n$ is quasi-coherent iff each \mathcal{S}_n is; similarly for a graded \mathcal{S} -module \mathcal{M} . The notations $\mathcal{S}^{(d)}$, $\mathcal{M}(n)$, etc., are used analogously to those for graded algebras and modules [see (2.1.1)].

Let $U = \text{Spec}(A) \subseteq Y$ be an open affine. Then $\mathcal{S}|_U = \tilde{S}$, where $S = \Gamma(U, \mathcal{S})$ is a graded A -algebra. Set $X_U = \text{Proj}(S)$. Given another affine $U' = \text{Spec}(A') \subseteq U$, we have a ring homomorphism $A \rightarrow A'$ corresponding to $U' \hookrightarrow U$, and the restriction homomorphism $S \rightarrow S' = \Gamma(U', \mathcal{S})$ is just the induced map $S \rightarrow S' = S \otimes_A A'$ (I, 1.6.4). Hence $X_{U'} = X_U \times_U U'$ by (2.8.10), i.e., $X_{U'} = f_U^{-1}(U')$, where $f_U: X_U \rightarrow U$ is the structure morphism. Let $\rho_{U',U}: X_{U'} \rightarrow X_U$ be the open immersion thus defined. Given $U'' \subseteq U' \subseteq U$, we have $\rho_{U'',U} = \rho_{U',U} \circ \rho_{U'',U'}$.

Proposition (3.1.2). — *Given a quasi-coherent sheaf of positively graded \mathcal{O}_Y -algebras \mathcal{S} , there is a prescheme $f: X \rightarrow Y$ over Y , unique up to canonical isomorphism, such that for every open affine $U \subseteq Y$, $f^{-1}(U)$ is identified with X_U , in such a way that for every $U' \subseteq U$, the inclusion $f^{-1}(U') \subseteq f^{-1}(U)$ is identified with $\rho_{U',U}$.*

(3.1.3). The prescheme X in (3.1.2) is the *homogeneous spectrum* of \mathcal{S} , denoted $\text{Proj}(\mathcal{S})$. X is separated over Y by (2.4.2) and (I, 5.5.5), and if \mathcal{S} is an \mathcal{O}_Y -algebra of finite type (I, 9.6.2), then X is of finite type over Y . For any open $U \subseteq Y$, we clearly have $f^{-1}(U) \cong \text{Proj}(\mathcal{S}|_U)$.

Proposition (3.1.4). — *Let $f \in \Gamma(Y, \mathcal{S}_d)$, $d > 0$. There is an open subset $X_f \subseteq X$ such that $X_f \cap X_U$ is the basic open set $D_+(f|_U)$ of X_U for each affine U . In particular, $X_f \cong \text{Spec}(\mathcal{S}^{(d)}/(f-1)\mathcal{S}^{(d)})$ is affine over Y .*

We call X_f the *non-vanishing locus* of f .

Corollary (3.1.5). — $X_{fg} = X_f \cap X_g$.

Corollary (3.1.6). — *If (f_α) is a family of homogeneous sections of \mathcal{S} , and if the sheaf of ideals in \mathcal{S} that they generate contains \mathcal{S}_n for all sufficiently large n , then the open sets X_{f_α} cover X .*

Corollary (3.1.7). — *If \mathcal{A} is a quasi-coherent sheaf of \mathcal{O}_Y -algebras, then $\text{Proj}(\mathcal{A}[t]) = \text{Spec}(\mathcal{A})$. In particular, $\text{Proj}(\mathcal{O}_Y[t]) = Y$.*

Proposition (3.1.8). — (i) $\text{Proj}(\mathcal{S}) \cong \text{Proj}(\mathcal{S}^{(d)})$ as a Y -scheme [see (2.4.7, (i))].

(ii) Let $\mathcal{S}' = \mathcal{O}_Y \oplus \bigoplus_{n>0} \mathcal{S}_n$. Then $\text{Proj}(\mathcal{S}) \cong \text{Proj}(\mathcal{S}')$ as a Y -scheme [see (2.4.8)].

(iii) Let \mathcal{L} be an invertible sheaf on Y , and let $\mathcal{S}_{(\mathcal{L})} = \bigoplus_n (\mathcal{S}_n \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes n})$. Then $\text{Proj}(\mathcal{S}) \cong \text{Proj}(\mathcal{S}_{(\mathcal{L})})$ as a Y -scheme.

(3.1.9). By (0, 4.1.3) and (I, 1.3.14), \mathcal{S}_1 generates \mathcal{S} iff $\Gamma(U_\alpha, \mathcal{S}_1)$ generates $\Gamma(U_\alpha, \mathcal{S})$, for all U_α in an affine covering; and if so, this holds for every affine $U \subseteq Y$.

Proposition (3.1.10). — Suppose Y has a finite affine open covering (U_i) such that each $\Gamma(U_i, \mathcal{S})$ is of finite type over $\Gamma(U_i, \mathcal{O}_Y)$. Then for some d , $\mathcal{S}^{(d)}$ is generated by \mathcal{S}_d , and \mathcal{S}_d is an \mathcal{O}_Y -module of finite type.

Corollary (3.1.11). — Under the hypotheses of (3.1.10), $\text{Proj}(\mathcal{S}) \cong \text{Proj}(\mathcal{S}')$, where \mathcal{S}' is generated as an \mathcal{O}_Y -algebra by \mathcal{S}'_1 , which is a finitely generated \mathcal{O}_Y -module.

(3.1.12). Let \mathcal{N} be the nilradical of \mathcal{S} . It's quasi-coherent by (I, 5.5.1). Put $\mathcal{N}_+ = \mathcal{N} \cap \mathcal{S}_+$, a graded \mathcal{S}_0 -module by (2.1.10). We call \mathcal{S} *essentially reduced* if $\mathcal{N}_+ = 0$, which is equivalent to \mathcal{S}_y being essentially reduced [see (2.1.10)] for all $y \in Y$. We call \mathcal{S} *integral* if \mathcal{S}_y is an integral domain with $(\mathcal{S}_y)_+ \neq 0$ for all $y \in Y$.

Proposition (3.1.13). — If $X = \text{Proj}(\mathcal{S})$, then $X_{\text{red}} \cong \text{Proj}(\mathcal{S}/\mathcal{N}_+)$. In particular, X is reduced if \mathcal{S} is essentially reduced [see (2.4.4, (i))].

Proposition (3.1.14). — Let Y be an integral prescheme, \mathcal{S} a graded quasi-coherent \mathcal{O}_Y -algebra such that $\mathcal{S}_0 = \mathcal{O}_Y$.

(i) If \mathcal{S} is integral (3.1.12), then $X = \text{Proj}(\mathcal{S})$ is integral, and the structure morphism $\phi: X \rightarrow Y$ is dominant.

(ii) Conversely, if \mathcal{S} is essentially reduced, X is integral, and ϕ is dominant, then \mathcal{S} is integral.

3.2. Sheaf on $\text{Proj}(\mathcal{S})$ associated to a graded \mathcal{S} -module.

(3.2.1). Let \mathcal{S} be a quasi-coherent sheaf of graded \mathcal{O}_Y -modules, \mathcal{M} a quasi-coherent sheaf of graded \mathcal{S} -modules (quasi-coherent as \mathcal{S} -module or equivalently as \mathcal{O}_Y -module). Keeping the notation of (3.1.1), let $\widetilde{\mathcal{M}}_U$ be the sheaf on X_U associated to $\Gamma(U, \mathcal{M})$ (2.5.3). If $U' \subseteq U$, then $\Gamma(U', \mathcal{M}) = \Gamma(U, \mathcal{M}) \otimes_A A'$, hence $\widetilde{\mathcal{M}}_{U'} = \rho_{U',U}^* \widetilde{\mathcal{M}}_U = \widetilde{\mathcal{M}}_U|_{X_{U'}}$.

Proposition (3.2.2). — There is a unique quasi-coherent \mathcal{O}_X -module $\widetilde{\mathcal{M}}$ such that $\widetilde{\mathcal{M}}|_{X_U} = \widetilde{\mathcal{M}}_U$ for all open affines $U \subseteq Y$.

Proposition (3.2.3). — Let $f \in \Gamma(Y, \mathcal{S}_d)$, $d > 0$. The isomorphism $X_f \cong \text{Spec}(\mathcal{S}^{(d)}/(f-1)\mathcal{S}^{(d)})$ identifies $\widetilde{\mathcal{M}}|_{X_f}$ with the sheaf associated to the $\mathcal{S}^{(d)}/(f-1)\mathcal{S}^{(d)}$ -module $\mathcal{M}^{(d)}/(f-1)\mathcal{M}^{(d)}$ [see (2.8.12)].

Proposition (3.2.4). — $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is an exact, covariant functor, which preserves direct sums and direct limits.

In particular, if $\mathcal{I} \subseteq \mathcal{S}$ is a homogeneous ideal sheaf, then $\widetilde{\mathcal{I}}$ is a sheaf of ideals in \mathcal{O}_X . If \mathcal{I} is a sheaf of ideals in \mathcal{O}_Y , then $(\mathcal{I}\mathcal{M})^\sim = \mathcal{I} \cdot \widetilde{\mathcal{M}}$.

Proposition (3.2.5). — Let $f \in \mathcal{S}_d$. The restriction of $\mathcal{S}(nd)^\sim$ to X_f is isomorphic to \mathcal{O}_{X_f} [with generating section f^n]. In particular, if \mathcal{S}_1 generates \mathcal{S} , then each $\mathcal{S}(n)^\sim$ is invertible [see (2.5.7-9)].

Proposition (3.2.6). — *There are canonical, functorial homomorphisms*

$$(3.2.6.1) \quad \lambda: \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{N}} \rightarrow (\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N})^\sim,$$

$$(3.2.6.2) \quad \mu: \mathcal{H}om_{\mathcal{S}}(\mathcal{M}, \mathcal{N})^\sim \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}).$$

If \mathcal{S}_1 generates \mathcal{S} , then λ is an isomorphism, and if in addition \mathcal{M} is finitely presented, then μ is an isomorphism [see (2.5.11-13)].

Corollary (3.2.7). — *[see (2.5.14)] If \mathcal{S}_1 generates \mathcal{S} , then for all $m, n \in \mathbb{Z}$,*

$$(3.2.7.1) \quad \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$$

$$(3.2.7.2) \quad \mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}.$$

Corollary (3.2.8). — *[see (2.5.15)] If \mathcal{S}_1 generates \mathcal{S} , then $(\mathcal{M}(n))^\sim = \widetilde{\mathcal{M}}(n)$.*

Remarks (3.2.9). — (i) If $\mathcal{S} = \mathcal{A}[t]$ as in (3.1.7), then $\mathcal{O}_X(n) \cong \mathcal{O}_X$ for all n . If \mathcal{N} is a quasi-coherent sheaf of \mathcal{A} -modules, $\mathcal{M} = \mathcal{N} \otimes_{\mathcal{A}} \mathcal{A}[t]$, then $\widetilde{\mathcal{M}}$ is the sheaf on $X \cong \text{Spec}(\mathcal{A})$ associated to \mathcal{N} as in (1.4.3).

(ii) If $\mathcal{S}'_0 = \mathcal{O}_Y$, $\mathcal{S}'_n = \mathcal{S}_n$ for $n > 0$, then the canonical isomorphism $X \cong X'$ identifies $\mathcal{O}_X(n)$ with $\mathcal{O}_{X'}(n)$. If $X^{(d)} = \text{Proj}(\mathcal{S}^{(d)})$, the canonical isomorphism $X \cong X^{(d)}$ identifies $\mathcal{O}_{X^{(d)}}(n)$ with $\mathcal{O}_X(nd)$ [see (2.5.16)].

Proposition (3.2.10). — *Let \mathcal{L} be an invertible sheaf on Y . The canonical isomorphism $X_{(\mathcal{L})} = \text{Proj}(\mathcal{S}_{(\mathcal{L})}) \cong X = \text{Proj}(\mathcal{S})$ in (3.1.8, (iii)) identifies $\mathcal{O}_{X_{(\mathcal{L})}}(n)$ with $\mathcal{O}_X(n) \otimes_Y \mathcal{L}^{\otimes n}$.*

3.3. Graded \mathcal{S} -module associated with a sheaf on $\text{Proj}(\mathcal{S})$. *In this section we assume that \mathcal{S}_1 generates \mathcal{S} . Recall that by (3.1.8, (i)) this restriction is inessential, given the finiteness conditions in (3.1.10).*

(3.3.1). Let $p: X = \text{Proj}(\mathcal{S}) \rightarrow Y$ be the structure morphism. For any sheaf of \mathcal{O}_X -modules \mathcal{F} , put

$$(3.3.1.1) \quad \Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{F}(n)).$$

In particular,

$$(3.3.1.2) \quad \Gamma_*(\mathcal{O}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{O}(n)).$$

The canonical homomorphism (0, 4.2.2) $p_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} p_*(\mathcal{G}) \rightarrow p_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$ makes $\Gamma_*(\mathcal{O})$ a graded \mathcal{O}_Y -algebra, and $\Gamma_*(\mathcal{F})$ a graded $\Gamma_*(\mathcal{O})$ -module. The functor $\Gamma_*(-)$ is left exact; in particular, if $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal sheaf, then $\Gamma_*(\mathcal{I})$ is a homogeneous ideal sheaf in $\Gamma_*(\mathcal{O})$.

(3.3.2). As in (2.6.2), for any graded \mathcal{S} -module \mathcal{M} , there is a canonical homomorphism of graded sheaves

$$(3.3.2.3) \quad \alpha: \mathcal{M} \rightarrow \Gamma_*(\widetilde{\mathcal{M}}).$$

In particular, $\alpha: \mathcal{S} \rightarrow \Gamma_*(\mathcal{O})$ is a homomorphism of sheaves of graded algebras, which makes $\Gamma_*(\widetilde{\mathcal{M}})$ a graded \mathcal{S} -module, and (??) an \mathcal{S} -module homomorphism.

Note that α_n induces $p^*(\mathcal{M}_n) \rightarrow \widetilde{\mathcal{M}}(n)$; this is the sheaf homomorphism associated by (3.2.4) to the canonical graded \mathcal{O}_Y -module homomorphism $\mathcal{M}_n \otimes_{\mathcal{O}_Y} \mathcal{S} \rightarrow \mathcal{M}(n)$.

Proposition (3.3.3). — *Given $f \in \Gamma(X, \mathcal{S}_d)$, $d > 0$, X_f is the non-vanishing locus of the section $\alpha_d(f)$ of the invertible sheaf $\mathcal{O}_X(d)$.*

(3.3.4). Suppose now that for every quasi-coherent \mathcal{F} on X , the sheaves $p_*(\mathcal{F})$ and hence $\Gamma_*(\mathcal{F})$ are quasi-coherent on Y . In particular, this holds if X is of finite type over Y (I, 9.2.2). Then $\Gamma_*(\mathcal{F})^\sim$ is defined and quasi-coherent on X . As in (2.6.4), there is a canonical homomorphism

$$(3.3.4.1) \quad \beta: \Gamma_*(\mathcal{F})^\sim \rightarrow \mathcal{F}.$$

Proposition (3.3.5). — *Let \mathcal{M} be a quasi-coherent graded \mathcal{S} -module, \mathcal{F} a quasi-coherent sheaf on \mathcal{O}_X . Then each of the following maps in the identity [see (2.6.5)]:*

$$(3.3.5.1) \quad \widetilde{\mathcal{M}} \xrightarrow{\tilde{\alpha}} \Gamma_*(\widetilde{\mathcal{M}})^\sim \xrightarrow{\beta} \widetilde{\mathcal{M}},$$

$$(3.3.5.2) \quad \Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\Gamma_*(\mathcal{F})^\sim) \xrightarrow{\Gamma_*^{(\beta)}} \Gamma_*(\mathcal{F}).$$

3.4. Finiteness conditions.

Proposition (3.4.1). — *Let \mathcal{S} be a quasi-coherent sheaf of graded \mathcal{O}_Y algebras, generated by \mathcal{S}_1 , and suppose further that \mathcal{S}_1 is an \mathcal{O}_Y -module of finite type. Then $X = \text{Proj}(\mathcal{S})$ is of finite type over Y [see (2.7.1, (ii))].*

(3.4.2). Consider two conditions on a graded \mathcal{S} -module \mathcal{M} :

(TF) There exists n such that $\bigoplus_{k \geq n} \mathcal{M}_k$ is a sheaf of \mathcal{S} -modules of finite type;

(TN) There exists n such that $\mathcal{M}_k = 0$ for $k \geq n$.

The terminology of (2.7.2) will be used in this context also.

Proposition (3.4.3). — *[see (2.7.3)] Assume that \mathcal{S}_1 is of finite type and generates \mathcal{S} .*

(i) *If \mathcal{M} satisfies (TF), then $\widetilde{\mathcal{M}}$ is of finite type.*

(ii) *If \mathcal{M} satisfies (TF), then $\widetilde{\mathcal{M}} = 0$ if and only if \mathcal{M} satisfies (TN).*

Theorem (3.4.4). — *Assume that \mathcal{S}_1 is of finite type and generates \mathcal{S} . For every quasi-coherent sheaf of \mathcal{O}_X -modules \mathcal{F} , the canonical homomorphism β in (3.3.4) is an isomorphism [see (2.7.5)].*

Corollary (3.4.5). — *Under the hypotheses of (3.4.4), every quasi-coherent \mathcal{O}_X -module \mathcal{F} is of the form $\widetilde{\mathcal{M}}$ for some graded \mathcal{S} -module \mathcal{M} [see (2.7.7)]. If \mathcal{F} is of finite type, and if Y is quasi-compact and separated, or if its underlying space is Noetherian, then \mathcal{M} can be taken to be of finite type [see (2.7.8)—the hypotheses on Y serve to imply that X is quasi-compact, by (3.4.1)].*

Corollary (3.4.6). — Under the hypotheses of (3.4.4), suppose further that Y is quasi-compact, and \mathcal{F} is of finite type. Then the canonical homomorphism $\sigma: p^*(p_*(\mathcal{F}(n))) \rightarrow \mathcal{F}(n)$ is surjective for all sufficiently large n .

Remarks (3.4.7). — For any morphism $p: X \rightarrow Y$ of ringed spaces, and \mathcal{O}_X -module \mathcal{F} , the surjectivity of $\sigma: p^*(p_*(\mathcal{F})) \rightarrow \mathcal{F}$ amounts to the following: for every $x \in X$ and every section s of \mathcal{F} on a neighborhood V of x , there is a neighborhood U of $p(x)$ in Y , a neighborhood $W \subseteq V \cap p^{-1}(U)$ of x , and finitely many sections $t_i \in \mathcal{F}(p^{-1}(U))$ and $a_i \in \mathcal{O}_X(W)$, such that

$$s|_W = \sum_i a_i(t_i|_W).$$

If Y is an affine scheme, and $p_*(\mathcal{F})$ is quasi-coherent, this is equivalent to \mathcal{F} being generated by its global sections on X . Hence for any morphism $p: X \rightarrow Y$ of preschemes, and any quasi-coherent \mathcal{O}_X -module \mathcal{F} such that $p_*(\mathcal{F})$ is quasi-coherent, the following are equivalent:

- (a) $\sigma: p^*(p_*(\mathcal{F})) \rightarrow \mathcal{F}$ is surjective;
- (b) there exists a quasi-coherent \mathcal{O}_Y -module \mathcal{G} such that $p^*(\mathcal{G}) \rightarrow \mathcal{F}$ is surjective;
- (c) for every open affine $U \subseteq Y$, $\mathcal{F}|_{p^{-1}(U)}$ is generated by its sections on $p^{-1}(U)$.

Corollary (3.4.8). — Under the hypotheses of (3.4.4), suppose that Y is quasi-compact and separated, or its underlying space is Noetherian. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Then for sufficiently large n , \mathcal{F} is isomorphic to a quotient of an \mathcal{O}_X -module of the form $(p^*(\mathcal{G}))(-n)$, where \mathcal{G} is a quasi-coherent \mathcal{O}_Y -module of finite type (depending on n).

3.5. Functorial behavior.

(3.5.1). Let $\phi: \mathcal{S}' \rightarrow \mathcal{S}$ be a homomorphism of graded quasi-coherent \mathcal{O}_Y -algebras, and set $X = \text{Proj}(\mathcal{S})$, $X' = \text{Proj}(\mathcal{S}')$, with structure morphisms $p: X \rightarrow Y$, $p': X' \rightarrow Y$. For each open affine $U \subseteq Y$, the homomorphism $\phi_U: \Gamma(U, \mathcal{S}') = S'_U \rightarrow S_U = \Gamma(U, \mathcal{S})$ induces a U -morphism $\Phi_U: G(\phi_U) \rightarrow X'_U$, by (2.8.1). For $V \subseteq U$, we have $G(\phi_V) = G(\phi_U) \cap p^{-1}(V)$, and Φ_V is the restriction of Φ_U to $G(\phi_V)$. Hence there is an open set $G(\phi) \subseteq X$ such that $G(\phi) \cap p^{-1}(U) = G(\phi_U)$ for every affine U , and a morphism $\Phi: G(\phi) \rightarrow X'$ whose restriction to $G(\phi_U)$ is Φ_U .

If every $y \in Y$ has a neighborhood U such that $\phi_U((S'_U)_+)$ generates $(S_U)_+$ [or more generally, such that the radical of the ideal it generates contains $(S_U)_+$], then $G(\phi) = X$.

Proposition (3.5.2). — (i) [see (2.8.7)] If \mathcal{M} is a quasi-coherent graded \mathcal{S} -module, then $(\mathcal{M}_{[\phi]})^\sim \cong \Phi_*(\widetilde{\mathcal{M}})$.

(ii) [see (2.8.8)] If \mathcal{M}' is a quasi-coherent graded \mathcal{S}' -module, there is a canonical functorial homomorphism $\Phi^*(\widetilde{\mathcal{M}'}) \rightarrow (\mathcal{M}' \otimes_{\mathcal{S}'} \mathcal{S})|_{G(\phi)}$. If \mathcal{S}'_1 generates \mathcal{S}' , it is an isomorphism.

In particular, for each n there is a canonical homomorphism

$$(3.5.2.1) \quad \Phi^*(\mathcal{O}_{X'}(n)) \rightarrow \mathcal{O}_X(n)|_{G(\phi)}.$$

Proposition (3.5.3). — Given a morphism $\psi: Y' \rightarrow Y$, and a quasi-coherent graded \mathcal{O}_Y -algebra \mathcal{S} , set $\mathcal{S}' = \psi^*\mathcal{S}$. Then $\text{Proj}(\mathcal{S}') \cong \text{Proj}(\mathcal{S}) \times_Y Y'$, and if \mathcal{M} is a quasi-coherent graded \mathcal{S} -module, then $\psi^*(\mathcal{M})^\sim \cong \widetilde{\mathcal{M}} \otimes_Y \mathcal{O}_{Y'}$.

Corollary (3.5.4). — In the setting of (3.5.3), $\mathcal{O}_{X'}(n) \cong \mathcal{O}_X(n) \otimes_Y Y'$, where $X' = \text{Proj}(\mathcal{S}')$, $X = \text{Proj}(\mathcal{S})$.

(3.5.5). Keeping the preceding notation, let $\Psi: X' \rightarrow X$ be the canonical morphism, and set $\mathcal{M}' = \psi^*(\mathcal{M})$. Assume that \mathcal{S}_1 generates \mathcal{S} and that X is of finite type over Y ; then the same hold for \mathcal{S}' , X' , Y' . Given an \mathcal{O}_X -module \mathcal{F} , set $\mathcal{F}' = \Psi^*(\mathcal{F})$. By (3.5.4) and (0, 4.3.3), we have $\mathcal{F}'(n) = \Psi^*(\mathcal{F}(n))$. Let

$$q: X \rightarrow Y, \quad q': X' \rightarrow Y'$$

be the structure morphisms. The canonical homomorphism $\mathcal{F}(n) \rightarrow \Psi_*(\Psi^*(\mathcal{F}(n))) = \Psi_*(\mathcal{F}'(n))$ gives rise to $q_*(\mathcal{F}(n)) \rightarrow q_*(\Psi_*(\mathcal{F}'(n))) = \psi_*(q'_*(\mathcal{F}'(n)))$. Hence we have a canonical Ψ -homomorphism $\theta: \Gamma_*(\mathcal{F}) \rightarrow \Gamma_*(\mathcal{F}')$. Then (2.8.13.1-2) yield commutative diagrams

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \beta_{\mathcal{F}} \uparrow & & \uparrow \beta_{\mathcal{F}'} \\ \Gamma_*(\mathcal{F})^\sim & \xrightarrow{\tilde{\theta}} & \Gamma_*(\mathcal{F}'), \\ \\ \Gamma_*(\widetilde{\mathcal{M}}) & \xrightarrow{\theta} & \Gamma_*(\widetilde{\mathcal{M}'}) \\ \alpha_{\mathcal{M}} \uparrow & & \uparrow \alpha_{\mathcal{M}'} \\ \mathcal{M} & \longrightarrow & \mathcal{M}' \end{array},$$

where the unlabelled horizontal arrows are the canonical Ψ - or ψ -morphisms.

(3.5.6). Now suppose given a morphism $g: Y' \rightarrow Y$, a graded quasi-coherent \mathcal{O}_Y -algebra (resp. $\mathcal{O}_{Y'}$ -algebra) \mathcal{S} (resp. \mathcal{S}'), and a g -homomorphism of graded algebras $u: \mathcal{S} \rightarrow \mathcal{S}'$ (i.e., a homomorphism $u: \mathcal{S} \rightarrow g_*(\mathcal{S}')$, or equivalently $u^\sharp: g^*(\mathcal{S}) \rightarrow \mathcal{S}'$). This gives a Y' -morphism $G(u^\sharp) \rightarrow \text{Proj}(g^*(\mathcal{S})) = X \times_Y Y'$, where $X = \text{Proj}(\mathcal{S})$, and $G(u^\sharp)$ is open in $X' = \text{Proj}(\mathcal{S}')$. Composing with the projection of $X \times_Y Y'$ on X , we get a morphism $v: G(u^\sharp) \rightarrow X$, denoted $v = \text{Proj}(u)$, and commutative diagram

$$\begin{array}{ccc} G(u^\sharp) & \xrightarrow{v} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{g} & Y. \end{array}$$

To any quasi-coherent graded \mathcal{S} -module \mathcal{M} there corresponds a canonical v -morphism

$$(3.5.6.1) \quad v: \widetilde{\mathcal{M}} \rightarrow (g^*(\mathcal{M}) \otimes_{g^*(\mathcal{S})} \mathcal{S}')^\sim | G(u^\sharp),$$

and if \mathcal{S}_1 generates \mathcal{S} , then v^\sharp is an isomorphism. In particular, we have

$$(3.5.6.2) \quad v: \mathcal{O}_X(n) \rightarrow \mathcal{O}_{X'}(n) | G(u^\sharp).$$

3.6. Closed subschemes of $\text{Proj}(\mathcal{S})$.

(3.6.1). Using (3.1.8), the analog of (2.9.1) holds for a homomorphism of graded quasi-coherent \mathcal{O}_Y -algebras $\phi: \mathcal{S} \rightarrow \mathcal{S}'$.

Proposition (3.6.2). — [see (2.9.2)] *Let $X = \text{Proj}(\mathcal{S})$.*

(i) *If $\phi: \mathcal{S} \rightarrow \mathcal{S}'$ is (TN)-surjective, then the associated morphism $\Phi = \text{Proj}(\phi)$ (3.5.1) is defined on all of $\text{Proj}(\mathcal{S}')$ and is a closed immersion into X . If $\mathcal{I} = \ker(\phi)$, the image of Φ is the closed subscheme defined by the ideal sheaf $\tilde{\mathcal{I}} \subseteq \mathcal{O}_X$.*

(ii) *Suppose further that $\mathcal{S}_0 = \mathcal{O}_Y$, \mathcal{S}_1 generates \mathcal{S} , and \mathcal{S}_1 is of finite type. Let $X' \subseteq X$ be a closed subscheme, defined by a quasi-coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$, and let $\mathcal{J} \subseteq \mathcal{S}$ be the preimage of $\Gamma_*(\mathcal{I})$ under $\alpha: \mathcal{S} \rightarrow \Gamma_*(\mathcal{O}_X)$ (3.3.2). Set $\mathcal{S}' = \mathcal{S}/\mathcal{J}$. Then X' is the image of the closed immersion $\text{Proj}(\mathcal{S}') \rightarrow X$ associated to the canonical surjection $\mathcal{S} \rightarrow \mathcal{S}'$.*

Corollary (3.6.3). — *In (3.6.2, (i)), if \mathcal{S}_1 generates \mathcal{S} , then $\Phi^*(\mathcal{O}_X(n)) = \mathcal{O}_{X'}(n)$ [see (2.9.3)].*

Corollary (3.6.4). — *Let \mathcal{S} be a quasi-coherent sheaf of graded \mathcal{O}_Y algebras such that \mathcal{S}_1 generates \mathcal{S} , let $u: \mathcal{M} \rightarrow \mathcal{S}_1$ be a surjective homomorphism of quasi-coherent \mathcal{O}_Y -modules, and let $\bar{u}: \mathbf{S}_{\mathcal{O}_Y}(\mathcal{M}) \rightarrow \mathcal{S}$ be the graded algebra homomorphism that extends u (1.7.4). Then the morphism $\text{Proj}(\bar{u})$ is a closed immersion of $\text{Proj}(\mathcal{S})$ into $\text{Proj}(\mathbf{S}_{\mathcal{O}_Y}(\mathcal{M}))$.*

3.7. Morphisms from a prescheme to a homogeneous spectrum.

(3.7.1). Let $q: X \rightarrow Y$ be a morphism of preschemes, \mathcal{L} an invertible \mathcal{O}_X -module, \mathcal{S} a graded quasi-coherent \mathcal{O}_Y -algebra; then $q^*(\mathcal{S})$ is a graded quasi-coherent \mathcal{O}_X -algebra. Suppose given a graded \mathcal{O}_X -algebra homomorphism

$$\psi: q^*(\mathcal{S}) \rightarrow \mathcal{S}' = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n},$$

or equivalently, a q -morphism of graded algebras

$$\psi^\flat: \mathcal{S} \rightarrow q_*(\mathcal{S}').$$

Now, $\text{Proj}(\mathcal{S}') = X$, by (3.1.7) and (3.1.8, (iii)), so we get an open subset $G(\psi) \subseteq X$ and a Y -morphism

$$(3.7.1.1) \quad r_{\mathcal{L}, \psi}: G(\psi) \rightarrow \text{Proj}(\mathcal{S}) = P$$

associated to \mathcal{L} and ψ , as in (3.5.6).

(3.7.2). Let us describe $r = r_{\mathcal{L}, \psi}$ more explicitly when $Y = \text{Spec}(A)$ is affine, so $\mathcal{S} = \tilde{S}$. First suppose $X = \text{Spec}(B)$ affine and $\mathcal{L} = \tilde{L}$, where L is a free B -module of rank 1, with generator c , say. Then ψ corresponds to a graded A -algebra homomorphism $S \otimes_A B \rightarrow B[c]$, necessarily of the form $(s \otimes b) \mapsto bv(s)c^n$ for $s \in S_n$, where $v: S \rightarrow B$ is an (ungraded) A -algebra homomorphism. Given $f \in S_d$, set $g = v(f)$. Then $r^{-1}(D_+(f)) = D(g)$, and the restriction $r: D(g) \rightarrow D_+(f)$ corresponds to the ring homomorphism $S_{(f)} \subseteq S_f \rightarrow B_g$ induced by v . Here $G(\psi)$ is the union of such open sets $D(g) \subseteq X$. The generalization to arbitrary X (Y still affine) is as follows.

Proposition (3.7.3). — If $Y = \text{Spec}(A)$ is affine and $\mathcal{S} = \widetilde{S}$, then for every $f \in S_d$, we have

$$(3.7.3.1) \quad r_{\mathcal{L},\psi}^{-1}(D_+(f)) = X_{\psi^b(f)} \quad (\text{where } \psi^b(f) \in \Gamma(X, \mathcal{L}^{\otimes d}))$$

and the restriction $X_{\psi^b(f)} \rightarrow D_+(f) = \text{Spec}(S_{(f)})$ corresponds (I, 2.2.4) to the algebra homomorphism

$$(3.7.3.2) \quad \psi_f^b: S_{(f)} \rightarrow \Gamma(X_{\psi^b(f)}, \mathcal{O}_X)$$

given, for $s \in S_{nd}$, by

$$(3.7.3.3) \quad \psi_f^b(s/f^n) = (\psi^b(s)|_{X_{\psi^b(f)}})/(\psi^b(f)|_{X_{\psi^b(f)}})^n.$$

Note that $G(\psi)$ is the union of the open sets $X_{\psi^b(f)}$ for $f \in S_d$, $d > 0$. We say that $r_{\mathcal{L},\psi}$ is defined everywhere if $G(\psi) = X$. This property is local with respect to Y .

Corollary (3.7.4). — Under the hypotheses of (3.7.3), $r_{\mathcal{L},\psi}$ is defined everywhere if and only if for every $x \in X$ there exists $d > 0$ and $s \in S_d$ such that $t = \psi^b(s) \in \Gamma(X, \mathcal{L}^{\otimes n})$ satisfies $t(x) \neq 0$.

This condition always holds if ψ is (TN)-surjective.

Similarly, the property that $r_{\mathcal{L},\psi}$ is dominant is local on Y , and for Y affine, we have:

Corollary (3.7.5). — Under the hypotheses of (3.7.3), $r_{\mathcal{L},\psi}$ is dominant if and only if for every $n > 0$, every $s \in S_n$ such that $\psi^b(s) \in \Gamma(X, \mathcal{L}^{\otimes n})$ is locally nilpotent, is itself nilpotent.

Proof: the condition says that if $r_{\mathcal{L},\psi}^{-1}(D_+(s))$ is empty, then $D_+(s)$ is empty [see (2.3.7)].

Proposition (3.7.6). — Given a morphism $q: X \rightarrow Y$, an invertible \mathcal{O}_X -module \mathcal{L} , quasi-coherent graded \mathcal{O}_Y -algebras $\mathcal{S}, \mathcal{S}'$, and algebra homomorphisms $u: \mathcal{S}' \rightarrow \mathcal{S}$, $\psi: q^*(\mathcal{S}) \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$, let $\psi' = \psi \circ q^*(u)$. If $r_{\mathcal{L},\psi'}$ is defined everywhere, then so is $r_{\mathcal{L},\psi}$. If u is (TN)-surjective and $r_{\mathcal{L},\psi'}$ is dominant, then so is $r_{\mathcal{L},\psi}$. Conversely, if u is (TN)-injective and $r_{\mathcal{L},\psi}$ is dominant, then so is $r_{\mathcal{L},\psi'}$.

Proposition (3.7.7). — Let Y be a quasi-compact prescheme, $q: X \rightarrow Y$ a quasi-compact morphism, \mathcal{L} an invertible \mathcal{O}_X -module, \mathcal{S} a quasi-coherent graded \mathcal{O}_Y -algebra, $\psi: q^*(\mathcal{S}) \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ an algebra homomorphism. Suppose \mathcal{S} is the inductive limit of a filtered system of quasi-coherent graded \mathcal{O}_Y -algebras (\mathcal{S}^λ) , and set $\psi_\lambda = \psi \circ q^*(\phi_\lambda)$, where $\phi_\lambda: \mathcal{S}^\lambda \rightarrow \mathcal{S}$ is the canonical homomorphism. Then $r_{\mathcal{L},\psi}$ is defined everywhere if and only if some $r_{\mathcal{L},\psi_\lambda}$ is defined everywhere; in that case $r_{\mathcal{L},\psi_\mu}$ is defined everywhere for all $\mu \geq \lambda$.

Corollary (3.7.8). — Under the hypotheses of (3.7.7), if the $r_{\mathcal{L},\psi_\lambda}$ are dominant, then so is $r_{\mathcal{L},\psi}$. The converse holds if the ϕ_λ are injective.

Remarks (3.7.9). — (i) With the notation of (3.7.1), there is a canonical homomorphism

$$(3.7.9.1) \quad \theta: r_{\mathcal{L},\psi}^*(\mathcal{O}_P(n)) \rightarrow \mathcal{L}^{\otimes n}.$$

defined as in (3.5.6.2).

(ii) Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Suppose q quasi-compact and separated, whence $q_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is quasi-coherent on Y . Then $\mathcal{M}' = \bigoplus_n \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is a quasi-coherent

graded \mathcal{S}' -module, and $\mathcal{M} = q_*(\mathcal{M}') = \bigoplus_n q_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is a quasi-coherent \mathcal{S} -module via ψ^b . There is a canonical \mathcal{O}_X -module homomorphism

$$(3.7.9.2) \quad \xi: r_{\mathcal{L},\psi}^*(\widetilde{\mathcal{M}}) \rightarrow \mathcal{F}|G(\psi).$$

3.8. Criteria for immersion into a homogeneous spectrum.

(3.8.1). With the notation of (3.7.1), the property that $r_{\mathcal{L},\psi}$ is an (open, closed) immersion is local on Y .

Proposition (3.8.2). — *Under the hypotheses of (3.7.3), $r_{\mathcal{L},\psi}$ is defined everywhere and is an immersion if and only if there exist sections $s_\alpha \in S_{n_\alpha}$ ($n_\alpha > 0$) such that, setting $f_\alpha = \psi^b(s_\alpha)$, the following hold:*

(i) *The open sets X_{f_α} cover X .*

(ii) *The X_{f_α} are affine.*

(iii) *For every α and every $t \in \Gamma(X_{f_\alpha}, \mathcal{O}_X)$, there exists $m > 0$ and $s \in S_{mn_\alpha}$ such that $t = (\psi^b(s)|X_{f_\alpha})/(f_\alpha|X_{f_\alpha})^m$.*

Moreover, $r_{\mathcal{L},\psi}$ is an open immersion if there exists (s_α) satisfying (i)-(iii) and:

(iv) *For every $m > 0$ and $s \in S_{mn_\alpha}$ such that $\psi^b(s)|X_{f_\alpha} = 0$, there exists k such that $s_\alpha^k s = 0$.*

Likewise, $r_{\mathcal{L},\psi}$ is a closed immersion if there exists (s_α) satisfying (i)-(iii) and:

(v) *The open sets $D_+(s_\alpha)$ cover $P = \text{Proj}(S)$.*

Corollary (3.8.3). — *Under the hypotheses of (3.7.6), if $r_{\mathcal{L},\psi'}$ is defined everywhere and is an immersion, then so is $r_{\mathcal{L},\psi}$. If in addition u is (TN)-surjective and $r_{\mathcal{L},\psi'}$ is an open (resp. closed) immersion, then so is $r_{\mathcal{L},\psi}$.*

Proposition (3.8.4). — *Assume the hypotheses of (3.7.7) and also that $q: X \rightarrow Y$ is of finite type. Then $r_{\mathcal{L},\psi}$ is defined everywhere and is an immersion if and only if the same holds for some $r_{\mathcal{L},\lambda}$, in which case it also holds for $r_{\mathcal{L},\mu}$, for all $\mu \geq \lambda$.*

Proposition (3.8.5). — *Assume that Y is quasi-compact and separated, or that its underlying space is Noetherian. Let $q: X \rightarrow Y$ be a morphism of finite type, \mathcal{L} an invertible \mathcal{O}_X -module, \mathcal{S} a quasi-coherent graded \mathcal{O}_Y -algebra, $\psi: \mathcal{S} \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ a graded algebra homomorphism. Then $r_{\mathcal{L},\psi}$ is defined everywhere and is an immersion if and only if there exist $n > 0$ and a sub- \mathcal{O}_Y -module $\mathcal{E} \subseteq \mathcal{S}_n$ of finite type such that:*

(a) *the homomorphism $\psi_n \circ q^*(j_n): q^*(\mathcal{E}) \rightarrow \mathcal{L}^{\otimes n}$ (where $j_n: \mathcal{E} \rightarrow \mathcal{S}_n$ is the inclusion) is surjective; and*

(b) *letting \mathcal{S}' be the (graded) sub- \mathcal{O}_Y -algebra of \mathcal{S} generated by \mathcal{E} , $j': \mathcal{S}' \rightarrow \mathcal{S}$ the inclusion, and $\psi' = \psi \circ q^*(j')$, $r_{\mathcal{L},\psi'}$ is defined everywhere and is an immersion.*

When these conditions hold, they also hold for every quasi-coherent sub- \mathcal{O}_Y -module $\mathcal{E} \subseteq \mathcal{E}' \subseteq \mathcal{S}_n$, and for the image of $\mathcal{E}^{\otimes k}$ in \mathcal{S}_{kn} .