Synopsis of material from EGA II
§2: Homogeneous prime spectra, 2.5–2.9

2.5 Sheaf associated to a graded module.

(2.5.1–2) *Proposition*: Given a graded S-module M, there is a unique quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ on $X = \mathrm{Proj}(S)$ such that $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ for every homogeneous $f \in S_+$, with restriction from $D_+(f)$ to $D_+(fg)$ given by the canonical homomorphism $M_{(f)} \to M_{(fg)}$.

(2.5.3) *Definition*: $\widetilde{M}$ in (2.5.2) is the *sheaf associated to the graded S-module M*.

(2.5.4) *Proposition*: $M \mapsto \widetilde{M}$ is an exact functor, which commutes with inductive limits and arbitrary direct sums.

(2.5.5) *Proposition*: For all $\mathfrak{p} \in \mathrm{Proj}(S)$, we have $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.

(2.5.6) *Proposition*: Suppose that for every $z \in M$ and every homogeneous $f \in S_+$, some power of f annihilates z. Then $\widetilde{M} = 0$. If S_1 generates S as an S_0-algebra, the condition is necessary and sufficient.

(2.5.7) *Proposition*: Let $f \in S_d$, $d > 0$. For every integer n, the sheaf $S(nd)\tilde{}\,|D_+(f)$ is isomorphic to $\mathcal{O}_X|D_+(f)$.

[Proof: multiplication by f^n gives an isomorphism of $S_{(f)}$-modules $S_{(f)} \cong S(nd)_{(f)}$.]

(2.5.8) *Corollary*: The restriction of $S(nd)\tilde{}$ to the open set $U = \bigcup_{f \in S_d} D_+(f)$ is invertible [*i.e.*, locally free of rank 1 (0, 5.4.1)].

(2.5.9) *Corollary*: If S_1 generates S, then $S(n)\tilde{}$ is an invertible sheaf on $X = \mathrm{Proj}(S)$, for every n.

(2.5.10) From now on, we denote $\mathcal{O}_X(n) = S(n)\tilde{}$ and $\mathcal{F}(n) = \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}$. If S_1 generates S, then $\mathcal{F} \to \mathcal{F}(n)$ is exact.

(2.5.11) If M, N are graded, there are canonical functorial homomorphisms

(2.5.11.1) $$\lambda_{(f)} \colon M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)},$$

and hence

(2.5.11.2) $$\lambda \colon \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to (M \otimes_S N)\tilde{}.$$

If $\mathcal{I}$ and $\mathcal{J}$ are graded ideals, then as $\widetilde{\mathcal{I}}$, $\widetilde{\mathcal{J}}$ are ideal sheaves, there is a canonical homomorphism $\widetilde{\mathcal{I}} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{J}} \to \mathcal{O}_X$, which is equal to the composite

(2.5.11.3) $$\widetilde{\mathcal{I}} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{J}} \xrightarrow{\lambda} (\mathcal{I} \otimes_S \mathcal{J})\tilde{} \to \mathcal{O}_X.$$

The two maps $\lambda \circ (\lambda \otimes 1)$ and $\lambda \circ (1 \otimes \lambda)$

(2.5.11.4) $$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \otimes_{\mathcal{O}_X} \widetilde{P} \to (M \otimes_S N \otimes_S P)\tilde{}$$

are the same.

(2.5.12) Similarly, we have

(2.5.12.1), $$\mu_{(f)} \colon \mathrm{Hom}_S(M, N)_{(f)} \to \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}),$$

since $M \mapsto M_{(f)}$ is a functor, and we deduce

$$\mu\colon \operatorname{Hom}_S(M,N)\tilde{} \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}).$$

Proofs of (2.5.11-12) are by reducing to $D_+(f)$ and using (I: 1.3.8,13).

(2.5.13) *Proposition*: Suppose S_1 generates S. Then λ in (2.5.11.2) is an isomorphism; and if M is finitely presented (2.1.1), then so is μ in (2.5.12.2). Moreover, if $\mathcal{I}$ is a homogeneous ideal, then $\widetilde{\mathcal{I}}\widetilde{M} = (\mathcal{I}M)\tilde{}$.

(2.5.14) *Corollary*: If S_1 generates S, then there are canonical isomorphisms $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$ for all integers m, n.

(2.5.15) *Corollary*: If S_1 generates S, then there is a canonical isomorphism $M(n)\tilde{} \cong \widetilde{M}(n)$, for every graded module M.

(2.5.16) Under the identifications $X = \operatorname{Proj}(S) \cong X' = \operatorname{Proj}(S') \cong X^{(d)} = \operatorname{Proj}(S^{(d)})$ of (2.4.7), we have $\mathcal{O}_X(n) \cong \mathcal{O}_{X'}(n)$ and $\mathcal{O}_{X^{(d)}}(n) \cong \mathcal{O}_X(nd)$.

(2.5.17) *Proposition*: The canonical homomorphisms $\mathcal{O}_X(nd)\otimes_{\mathcal{O}}\mathcal{O}_X(md) \to \mathcal{O}_X((m+n)d)$ restrict to isomorphisms on $U = \bigcup_{f\in S_d} D_+(f)$.

2.6 Graded S-module associated to a sheaf on $\operatorname{Proj}(S)$.

In this section we assume that S_1 generates the ideal S_+, and write $X = \operatorname{Proj}(S)$. Then $\mathcal{O}_X(1)$ is invertible.

(2.6.1) Given an $\mathcal{O}_X$-module $\mathcal{F}$, define (0, 5.4.6)

$$\Gamma_*(\mathcal{F}) = \Gamma_*(\mathcal{O}_X(1),\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X,\mathcal{F}(n)). \tag{2.6.1.1}$$

Note (2.5.14) for the second equality above. Then $\Gamma_*(\mathcal{O}_X)$ is a graded ring and $\Gamma_*(\mathcal{F})$ is a graded $\Gamma_*(\mathcal{O}_X)$-module. Since $\mathcal{O}_X(n)$ is locally free, $\mathcal{F} \to \Gamma_*(\mathcal{F})$ is a *left exact* functor. In particular, if $\mathcal{I}$ is an ideal sheaf, then $\Gamma_*(\mathcal{I})$ is a homogeneous ideal in $\Gamma_*(\mathcal{O}_X)$.

(2.6.2) The map $M_0 \to M_{(f)}$, $x \mapsto x/1$, induces maps $M_0 \to \Gamma(D_+(f),\widetilde{M})$ for all homogeneous $f \in S_+$, compatible with restricions, and hence a map

$$\alpha_0\colon M_0 \to \Gamma(X,\widetilde{M}). \tag{2.6.2.1}$$

Applying this to $M(n)$ and using (2.5.15), get

$$\alpha_n\colon M_n = M(n)_0 \to \Gamma(X,\widetilde{M}(n)), \tag{2.6.2.2}$$

and hence a homomorphism of graded abelian groups

$$\alpha\colon M \to \Gamma_*(\widetilde{M}). \tag{2.6.2.3}$$

The map $\alpha\colon S \to \Gamma_*(\mathcal{O}_X)$ is a graded ring homomorphism, and (2.6.2.3) is an S-module homomorphism.

(2.6.3) *Proposition*: For every $f \in S_d$ ($d > 0$), the open set $D_+(f)$ is the non-vanishing locus of the section $\alpha(f)$ of $\mathcal{O}_X(d)$ (0, 5.5.2).

(2.6.4) Set $M = \Gamma_*(\mathcal{F})$, which we may consider as an S-module via $S \to \Gamma_*(\mathcal{O}_X)$. By (2.6.3), the section $\alpha_d(f)$ of $\mathcal{O}_X(d)$ is invertible on $D_+(f)$. Hence we can define an $S_{(f)}$-module homomorphism $\beta_{(f)}\colon M_{(f)} \to \Gamma(D_+(f), \mathcal{F})$ by $z/f^n \mapsto (z|D_+(f))/(\alpha_d(f)|D_+(f))^n$. This is compatible with restriction to $D_+(fg)$, giving a canonical $\mathcal{O}_X$-module homomorphism

(2.6.4.2) $$\beta\colon \Gamma_*(\mathcal{F})\tilde{} \to \mathcal{F}.$$

(2.6.5) *Proposition*: For any graded S-module M and sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, each of the following maps is the identity:

$$\widetilde{M} \xrightarrow{\tilde{\alpha}} \Gamma_*(\widetilde{M})\tilde{} \xrightarrow{\beta} \widetilde{M},$$

$$\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\Gamma_*(\mathcal{F})\tilde{}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F})$$

2.7 Finiteness conditions.

(2.7.1) *Proposition*: (i) If S is a Noetherian graded ring, then $X = \mathrm{Proj}(S)$ is a Noetherian scheme.

(ii) If S is a finitely-generated graded A-algebra, then X is a scheme of finite type over $Y = \mathrm{Spec}(A)$.

(2.7.2) Consider two conditions on a graded S-module M:

(TF) There exists n such that $\bigoplus_{k \geq n} M_k$ is a finitely generated S-module;

(TN) There exists n such that $M_k = 0$ for $k \geq n$.

A graded S-module homomorphism u will be called (TN)-*injective* (resp. (TN)-*surjective*, (TN)-*bijective*) if its kernel (resp. cokernel, both) satisfies (TN). By (2.5.4), this implies that $\widetilde{u}$ is injective (resp. surjective, bijective).

(2.7.3) *Proposition*: Assume that S_+ is a finitely generated ideal.

(i) If M satisfies (TF), then $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite type.

(ii) If M satisfies (TF), then $\widetilde{M} = 0$ if and only if M satisfies (TN).

(2.7.4) *Corollary*: If S_+ is finitely generated, then $\mathrm{Proj}(S) = \emptyset$ iff there is an n such that $S_k = 0$ for all $k \geq n$.

(2.7.5) *Theorem*: Let $X = \mathrm{Proj}(S)$, where S_+ is generated by finitely many elements, homogeneous of degree 1. Then for every quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the canonical homomorphism $\beta\colon \Gamma_*(\mathcal{F})\tilde{} \to \mathcal{F}$ is an isomorphism.

(2.7.6) *Remark*: If S is Noetherian and S_1 generates S_+, then the hypotheses of (2.7.5) hold.

(2.7.7) *Corollary*: Under the hypotheses of (2.7.5), every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is isomorphic to $\widetilde{M}$ for some graded S-module M.

(2.7.8) *Corollary*: Under the hypotheses of (2.7.5), every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type is isomorphic to $\widetilde{N}$ for some finitely generated graded S-module N.

[Proof: let $\mathcal{F} \cong \widetilde{M}$. Then $\mathcal{F}$ is the direct limit of its subsheaves $\widetilde{N}$ for $N \subseteq M$ finitely generated, by (2.5.4). Since $\mathcal{F}$ is of finite type and X is quasi-compact, the result follows by (0, 5.2.3).]

(2.7.9) *Corollary*: Under the hypotheses of (2.7.5), let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Then there exists n_0 such that for all $n \geq n_0$, $\mathcal{F}(n)$ is isomorphic to a quotient of $\mathcal{O}_X^k$ (where k depends on n), *i.e.*, $\mathcal{F}(n)$ is generated by finitely many global sections (0, 5.1.1).

(2.7.10) *Corollary*: Under the hypotheses of (2.7.5), let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Then there exists n_0 such that for all $n \geq n_0$, $\mathcal{F}$ is isomorphic to a quotient of $\mathcal{O}_X(-n)^k$ (where k depends on n).

(2.7.11) *Proposition*: Assume the hypotheses of (2.7.5) hold, and let M be a graded S-module.

(i) The canonical homomorphism $\widetilde{\alpha}\colon \widetilde{M} \to \Gamma_*(\widetilde{M})\widetilde{\ }$ is an isomorphism.

(ii) Let $\mathcal{G} \subseteq \widetilde{M}$ be a quasi-coherent sub-$\mathcal{O}_X$-module sheaf, and let $N \subseteq M$ be the preimage of $\Gamma_*(\mathcal{G}) \subseteq \Gamma_*(\widetilde{M})$ via α. Then $\widetilde{N} = \mathcal{G}$.

2.8 Functorial behavior.

(2.8.1) Let $\phi\colon S' \to S$ be a graded ring homomorphism. Denote the complement of $V_+(\phi(S'_+))$ in $X = \operatorname{Proj}(S)$ by $G(\phi)$. Equivalently, $G(\phi)$ is the union of the open sets $D_+(\phi(f'))$ for homogeneous $f' \in S_+$. Then ${}^a\phi\colon \operatorname{Spec}(S) \to \operatorname{Spec}(S')$ induces a continuous map ${}^a\phi\colon G(\phi) \to \operatorname{Proj}(S')$, such that

$$(2.8.1.1). \qquad {}^a\phi^{-1}(D_+(f')) = D_+(\phi(f')).$$

Let $f = \phi(f')$. Then ϕ induces $\phi_f\colon S'_{f'} \to S_{\phi(f')}$ and $\phi_{(f)}\colon S'_{(f')} \to S_{(\phi(f'))}$, hence a morphism ${}^a\phi_{(f)}\colon D_+(f) \to D_+(f')$, which on the underlying space is the restriction of ${}^a\phi$ to the open sets in (2.8.1.1). These are compatible with restriction to $D_+(fg)$.

(2.8.2) *Proposition*: There is a unique morphism $({}^a\phi, \widetilde{\phi})\colon G(\phi) \to \operatorname{Proj}(S')$ (called the morphism associated to ϕ and denoted $\operatorname{Proj}(\phi)$) whose restriction to each $D_+(\phi(f'))$ coincides with ${}^a\phi_{(f)}$.

(2.8.3) *Corollary*: (i) $\operatorname{Proj}(\phi)$ is an affine morphism.

(ii) If $\ker(\phi)$ is nilpotent (in particular, if ϕ is injective), then $\operatorname{Proj}(\phi)$ is dominant.

In general there exist morphisms $\operatorname{Proj}(S) \to \operatorname{Proj}(S')$ which are not affine, hence are not of the form $\operatorname{Proj}(\phi)$, for example $\operatorname{Proj}(S) \to \operatorname{Spec}(A) = \operatorname{Proj}(A[t])$ when S is an A-algebra.

(2.8.4) Given a third ring S'' and $\phi'\colon S'' \to S'$, let $\phi'' = \phi \circ \phi'$. Then $G(\phi'') \subseteq G(\phi)$, and if Φ, Φ', Φ'' are the associated morphisms, then $\Phi'' = \Phi' \circ (\Phi|G(\phi''))$.

(2.8.5) Suppose S (resp. S') is a graded A-algebra (resp. A'-algebra), and $\psi\colon A' \to A$ commutes with $\phi\colon S' \to S$. Then $G(\phi)$ and $\operatorname{Proj}(S')$ are schemes over $\operatorname{Spec}(A)$, $\operatorname{Spec}(A')$ respectively, and the the corresponding diagram commutes.

(2.8.6) Let M be a graded S-module, which we may consider as a graded S'-module $M_{[\phi]}$.

(2.8.7) *Proposition*: There is a canonical functorial isomorphism $(M_{[\phi]})\widetilde{\ } \cong \Phi_*(\widetilde{M}|G(\phi))$, where $\Phi = \operatorname{Proj}(\phi)$.

(2.8.8) *Proposition*: Let M' be a graded S'-module. There is a canonical functorial homomorphism $\nu\colon \Phi^*(\widetilde{M'}) \to (M' \otimes_{S'} S)\widetilde{\ }|G(\phi)$. If S'_1 generates S', then ν is an isomorphism.

(2.8.9) Let $\psi\colon A' \to A$ be a ring homomorphism, $\Psi\colon Y = \operatorname{Spec}(A) \to \operatorname{Spec}(A') = Y'$ its associated morphism. Let S' be a positively graded A'-algebra; then $S = S' \otimes_{A'} A$ is a positively graded A-algebra. We have the ring homomorphism $\phi\colon S' \to S$, $\phi(s') = s' \otimes 1$, and $\phi(S'_+)$ generates S_+ as an A-module, hence $G(\phi) = \operatorname{Proj}(S) = X$. Set $X' = \operatorname{Proj}(S')$. Further, let M' be a graded S'-module, and set $M = M' \otimes_{A'} A = M' \otimes_{S'} S$.

(2.8.10) *Proposition*: With the notation of (2.8.9), we have $X = X' \times_{Y'} Y$, and the canonical homomorphism $\nu\colon \Phi^*(\widetilde{M'}) \to \widetilde{M}$ (2.8.8) is an isomorphism.

(2.8.11) *Corollary*: For all $n \in \mathbb{Z}$, $\widetilde{M}(n)$ is identified with $\Phi^*(\widetilde{M'}(n)) = \widetilde{M'}(n) \otimes_{Y'} \mathcal{O}_Y$. In particular, $\mathcal{O}_X(n) = \Phi^*\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(n) \otimes_{Y'} \mathcal{O}_Y$.

(2.8.12) For $f' \in S'_d$ $(d > 0)$ and $f = \phi(f')$, the canonical map $M'_{(f')} \to M_{(f)}$ is identified with $M'^{(d)}/(f'-1)M'^{(d)} \to M^{(d)}/(f-1)M^{(d)}$ by (2.2.5).

(2.8.13) Keep the setting of (2.8.9). Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module, and set $\mathcal{F} = \Phi^*(\mathcal{F}')$. Then $\mathcal{F}(n) = \Phi^*(\mathcal{F}'(n))$ by (2.8.11) and (0, 4.3.3). From (0, 4.4.1) we have $\Gamma(\rho)\colon \Gamma(X', \mathcal{F}'(n)) \to \Gamma(X, \mathcal{F}(n))$ for all $n \in \mathbb{Z}$, giving a homomorphism of graded modules

$$\Gamma_*(\mathcal{F}') \to \Gamma_*(\mathcal{F}).$$

If S_1 generates S, and $\mathcal{F}' = \widetilde{M'}$, then $\mathcal{F} = \widetilde{M}$, where $M = M' \otimes_{A'} A$, and we have a commutative diagram

(2.8.13.1)
$$\begin{array}{ccc} M' & \xrightarrow{\alpha_{M'}} & \Gamma_*(\widetilde{M'}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha_M} & \Gamma_*(\widetilde{M}). \end{array}$$

Similarly, we have

(2.8.13.2)
$$\begin{array}{ccc} \Gamma_*(\mathcal{F}')\widetilde{\ } & \xrightarrow{\beta_{\mathcal{F}'}} & \mathcal{F}' \\ \downarrow & & \rho\downarrow \\ \Gamma_*(\mathcal{F})\widetilde{\ } & \xrightarrow{\beta_{\mathcal{F}}} & \mathcal{F}, \end{array}$$

in which the vertical arrows are Φ-morphisms.

(2.8.14) Given a second graded S'-module N', we have a canonical homomorphism

(2.8.14.1)
$$\Phi^*((M' \otimes_{S'} N')\widetilde{\ }) \to (M \otimes_S N)\widetilde{\ },$$

and a commutative diagram

(2.8.14.2)
$$\begin{array}{ccc} \Phi^*(\widetilde{M'} \otimes_{\mathcal{O}_{X'}} \widetilde{N'}) & \xrightarrow{\sim} & \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \\ {\scriptstyle \Phi^*(\lambda)}\downarrow & & \lambda\downarrow \\ \Phi^*((M' \otimes_{S'} N')\widetilde{\ }) & \longrightarrow & (M \otimes_S N)\widetilde{\ }, \end{array}$$

where the top row is the canonical isomorphism (0, 4.3.3). If S'_1 generates S', then also S_1 generates S, the vertical arrows are isomorphisms by (2.5.13), and hence (2.8.14.1) is an isomorphism.

Similarly, there is a commutative diagram

$$\begin{array}{ccc} \Phi^*(\mathrm{Hom}_{S'}(M',N')\tilde{\ }) & \longrightarrow & \mathrm{Hom}_S(M,N)\tilde{\ } \\ {\scriptstyle \Phi^*(\mu)}\big\downarrow & & {\scriptstyle \mu}\big\downarrow \\ \Phi^*(\mathcal{H}om_{\mathcal{O}_{X'}}(\widetilde{M'},\widetilde{N'})) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}), \end{array}$$

with bottom row given by (0, 4.4.6) and vertical arrows by (2.5.12).

(2.8.15) One can replace S_0 and S'_0 by $\mathbb{Z}$, or replace S and S' by $S^{(d)}$ and $S'^{(d)}$ as in (2.4.7), without changing Φ.

2.9 Closed subschemes of $\mathrm{Proj}(S)$**.**

(2.9.1) If $\phi\colon S' \to S$ is (TN)-*injective* (resp. (TN)-surjective, (TN)-bijective) (2.7.2), then (2.8.15) shows that where Φ is concerned we can reduce to the case that ϕ is actually injective (resp. surjective, bijective).

(2.9.2) *Proposition*: Let $X = \mathrm{Proj}(S)$.

(i) If $\phi\colon S \to S'$ is (TN)-surjective, then the associated morphism Φ is defined on all of $\mathrm{Proj}(S')$ and is a closed immersion into X. If $\mathcal{I} = \ker(\phi)$, the image of Φ is the closed subscheme defined by the ideal sheaf $\widetilde{\mathcal{I}}$.

(ii) Suppose further that S_+ is generated by finitely many elements, homogeneous of degree 1. Let $X' \subseteq X$ be a closed subscheme, defined by a quasi-coherent sheaf of ideals $\mathcal{J}$, and let $\mathcal{I} \subseteq S$ be the preimage of $\Gamma_*(\mathcal{J})$ under $\alpha\colon S \to \Gamma_*(\mathcal{O}_X)$ (2.6.2). Set $S' = S/\mathcal{I}$. Then X' is the image of the closed immersion $\mathrm{Proj}(S') \to X$ associated to the canonical surjection $S \to S'$.

(2.9.3) *Corollary*: In (2.9.2 (i)), if S_1 generates S_+, then $\Phi^*(S(n)\tilde{\ }) = S'(n)\tilde{\ }$ for all n, and $\Phi^*(\mathcal{F}(n)) = (\Phi^*(\mathcal{F}))(n)$ for every $\mathcal{O}_X$-module $\mathcal{F}$.

(2.9.4) In (2.9.2 (ii)), the subscheme X' is integral iff the ideal $\mathcal{I}$ is prime.

["If" is clear from (2.4.4). "Only if" uses (I, 7.4.4).]

(2.9.5) Let S be an A-algebra generated by S_1, M an A-module, and $u\colon M \to S_1$ a surjective A-module homomorphism, inducing $\overline{u}\colon \mathbf{S}(M) \to S$, where $\mathbf{S}(M)$ is the symmetric algebra. Then $\overline{u}$ induces a closed immersion of $\mathrm{Proj}(S)$ into $\mathrm{Proj}(\mathbf{S}(M))$.