

SYNOPSIS OF MATERIAL FROM EGA II
§2: HOMOGENEOUS PRIME SPECTRA, 2.1–2.4

2.1 Generalities on graded rings and modules.

(2.1.1) *Notation:* $S = \bigoplus_{n \geq 0} S_n$ a graded ring (in positive degrees), $S_+ = \bigoplus_{n > 0} S_n$. S_0 is a subring; S_+ is an ideal. $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a graded module; each M_n is an S_0 -module. $S^{(d)} = \bigoplus_n S_{nd}$ ($d > 0$) is a graded subring. $M^{(d,k)} = \bigoplus_n M_{nd+k}$ is a graded $S^{(d)}$ -module. We write $M^{(d)}$ for $M^{(d,0)}$.

$M(n)$ is the degree-shifted module $(M(n))_k = M_{n+k}$. In particular, defining $S_n = 0$ for $n < 0$, $S(n)$ is a graded S -module. A *free* graded module is direct sum of modules of the form $S(n)$. A free graded module has a homogeneous basis, since $S(n)$ is cyclic with generator in degree $-n$. M is *finitely presented* if there is an exact sequence $P \rightarrow Q \rightarrow M$, where P, Q are finitely-generated free graded modules, and the maps are homogeneous of degree zero.

(2.1.2) $M \otimes_S N$ is graded, with $(M \otimes_S N)_n$ the image of $\bigoplus_{p+q=n} M_p \otimes_{S_0} N_q$. Let H_n be the set of S -module homomorphisms $M \rightarrow N$ homogeneous of degree n . We set $\text{Hom}_S(M, N) = \bigoplus_n H_n$; it is a graded S -module. In general, $\text{Hom}_S(M, N)$ is strictly smaller than the module of all S -module homomorphisms $M \rightarrow N$, although they are equal if M is finitely generated. A *homomorphism of graded S -modules* from M to N is an element of $(\text{Hom}_S(M, N))_0$. Note that $M(m) \otimes_S N(n) = (M \otimes_S N)(m+n)$ and $\text{Hom}_S(M(m), N(n)) = (\text{Hom}_S(M, N))(n-m)$.

If $\phi: S \rightarrow S'$ is a ring homomorphism homogeneous of degree 0, it makes S' a *graded S -algebra*. Then $M \otimes_S S'$ is naturally a graded S' -module.

(2.1.3) *Lemma:* A set of homogeneous elements $E \subseteq S_+$ generates S_+ as an ideal in S iff E generates S as an S_0 -algebra.

(2.1.4) *Corollary:* S_+ is a finitely generated ideal iff S is a finitely generated S_0 -algebra.

(2.1.5) *Corollary:* S is Noetherian iff S_0 is Noetherian and S is a finitely generated S_0 -algebra.

(2.1.6) *Lemma:* Let S be positively graded and finitely generated as an S_0 -algebra. Let M be a finitely generated graded S -module. Then:

- (i) Each M_n is a finitely generated S_0 -module, and $M_n = 0$ for all $n < n_0$, for some n_0 .
- (ii) For some n_1 and h , we have $M_{n+h} = S_h M_n$ for all $n \geq n_1$.
- (iii) For every $d > 0$ and $0 \leq k < d$, $M(d, k)$ is a finitely generated $S^{(d)}$ -module.
- (iv) For every $d > 0$, $S^{(d)}$ is a finitely generated S_0 -algebra.
- (v) There exists $h > 0$ such that $S_{mh} = S_m^h$ for all $m > 0$.
- (vi) For every $n > 0$, there exists m_0 such that $S_m \subseteq (S_+)^n$ for all $m \geq m_0$.

(2.1.7) *Corollary:* If S is Noetherian, so is every $S^{(d)}$.

(2.1.8) Let $\mathfrak{p} = \bigoplus_n (\mathfrak{p} \cap S_n)$ be a homogeneous prime ideal of S . Suppose $\mathfrak{p} \not\supseteq S_+$. If $f \in S_+ \setminus \mathfrak{p}$, we have $f^n x \in \mathfrak{p}$ iff $x \in \mathfrak{p}$. In particular, if $f \in S_d$ ($d > 0$), and $x \in S_{m-nd}$, then $f^n x \in \mathfrak{p}_m$ iff $x \in \mathfrak{p}_{m-nd}$.

(2.1.9) *Proposition:* Let $n_0 > 0$, and for all $n \geq n_0$ let \mathfrak{p}_n be a subgroup of S_n . For there to exist a homogeneous prime ideal $\mathfrak{p} \not\supseteq S_+$ such that $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ for all $n \geq n_0$, it is necessary and sufficient that the following hold:

- (i) $S_m \mathfrak{p}_n \subseteq \mathfrak{p}_{m+n}$ for all $n \geq n_0$ and $m \geq 0$.
- (ii) For all $m, n \geq n_0$, $f \in S_m$, $g \in S_n$, $fg \in \mathfrak{p}_{m+n}$ implies $f \in \mathfrak{p}_m$ or $g \in \mathfrak{p}_n$.
- (iii) $\mathfrak{p}_n \neq S_n$ for some $n \geq n_0$.

(2.1.10) An ideal \mathcal{I} of S , $\mathcal{I} \subseteq S_+$ is called an *ideal of S_+* , and a *homogeneous prime ideal of S_+* if $\mathcal{I} = \mathfrak{p} \cap S_+$ for a homogeneous prime ideal $\mathfrak{p} \not\supseteq S_+$. Then \mathfrak{p} is unique by (2.1.8). The *radical in S_+* of an ideal \mathcal{I} of S_+ is $\sqrt{+\mathcal{I}} \stackrel{\text{def}}{=} (\sqrt{\mathcal{I}}) \cap S_+$; the *nilradical of S_+* is $\mathfrak{N}_+ = \sqrt{+}0$. S is called *essentially reduced* if $\mathfrak{N}_+ = 0$.

If \mathcal{I} is homogeneous, then so is $\sqrt{+\mathcal{I}}$: reducing to the case $\mathcal{I} = 0$, we must show that if $x = x_h + x_{h+1} + \cdots + x_k$ is nilpotent, then so is each x_i . The highest homogeneous component of x^n is x_k^n , so x_k is nilpotent; now replace x by $x - x_k$ and proceed by induction.

(2.1.11) If $x \in S$ is a zero-divisor, then so is its highest homogeneous component. S is *essentially integral* if S_+ (considered as a ring without unit) has no zero-divisors $\neq 0$. It suffices that it has no homogeneous zero-divisors $\neq 0$. If \mathfrak{p} is a homogeneous prime ideal of S_+ , then S/\mathfrak{p} is essentially integral.

Suppose S essentially integral. If $x_0 \in S_0$, and $x_0 f = 0$ for some homogeneous $f \in S_+$, it follows that $x_0 S_+ = 0$. Hence S is an integral domain if and only if S_0 is an integral domain and the annihilator of S_+ is zero.

2.2 Rings of fractions of a graded ring.

(2.2.1) If S is graded, $f \in S_d$, $d > 0$, then S_f is \mathbb{Z} -graded, with $(S_f)_n = \{x/f^k : x \in S_{n+kd}\}$. We denote $S_{(f)} = (S_f)_0$. The monomials $(f/1)^h$ ($h \in \mathbb{Z}$) are a basis of $(S^{(d)})_f$ as a free module over $S_{(f)}$; thus $(S^{(d)})_f \cong S_{(f)}[t, t^{-1}]$.

If M is a graded S -module, then M_f is a graded S_f -module, with $(M_f)_n = \{z/f^k : z \in M_{n+kd}\}$. We denote $M_{(f)} = (M_f)_0$, an $S_{(f)}$ -module. Then $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

(2.2.2) *Lemma:* Let $f \in S_d$, $g \in S_e$, $d, e > 0$. There is a canonical isomorphism of rings $S_{(fg)} \cong (S_{(f)})_{g^d/f^e}$, and after identifying them, a canonical isomorphism of modules $M_{(fg)} \cong (M_{(f)})_{g^d/f^e}$.

(2.2.3) The canonical homomorphism $S_{(f)} \rightarrow (S_{(f)})_{g^d/f^e} \cong S_{(fg)}$ is the degree-zero part of $S_f \rightarrow S_{fg}$, mapping x/f^k to $g^k x / (fg)^k$. Similarly for modules.

(2.2.4) *Lemma:* For $f, g \in S_+$ homogeneous, the ring $S_{(fg)}$ is generated by the images of the canonical homomorphisms from $S_{(f)}$ and $S_{(g)}$.

(2.2.5) *Proposition:* Let $f \in S_d$, $d > 0$. There is a canonical isomorphism of rings $S_{(f)} \cong S^{(d)}/(f-1)S^{(d)}$, and after identifying them, a canonical isomorphism of modules $M_{(f)} \cong M^{(d)}/(f-1)M^{(d)}$.

(2.2.6) *Corollary:* If S is Noetherian, so is $S_{(f)}$ for all f homogeneous of degree > 0 .

(2.2.7) If T is a multiplicative set of homogeneous elements in S_+ , then $T_0 = T \cup \{1\}$ is a multiplicative set in S ; $T_0^{-1}S$ is graded in the obvious way; we set $S_{(T)} = (T_0^{-1}S)_0$. Since

$T_0^{-1}S$ is the inductive limit of the rings S_f , as f ranges over T_0 , therefore $S_{(T)}$ is the inductive limit of the rings $S_{(f)}$. Defining $M_{(T)}$ similarly, it is the inductive limit of the modules $M_{(f)}$. If \mathfrak{p} is a homogeneous prime ideal of S_+ , then $S_{(\mathfrak{p})}$, $M_{(\mathfrak{p})}$ stand for $S_{(T)}$, $M_{(T)}$, where T is the set of homogeneous elements in $S_+ \setminus \mathfrak{p}$.

2.3 Homogeneous prime spectrum of a graded ring. [Liu, §2.3.3]

(2.3.1) Given a positively graded ring S , its *homogeneous prime spectrum* $\text{Proj}(S)$ is the set of homogeneous prime ideals of S_+ (2.1.10), that is, the set of homogeneous prime ideals of S not containing S_+ . We will define a scheme whose underlying set is $\text{Proj}(S)$.

(2.3.2–3) For any subset $E \subseteq S$, define $V_+(E)$ to be the set of homogeneous primes of S such that $E \subseteq \mathfrak{p} \not\subseteq S_+$, that is, $V_+(E) = V(E) \cap \text{Proj}(S) \subseteq \text{Spec}(S)$.

(2.3.2.1–3) From (I, 1.1.2), we see that the sets $V_+(E)$ are the closed sets of the subspace topology on $\text{Proj}(S) \subseteq \text{Spec}(S)$.

(2.3.2.4) Replacing E by the ideal \mathcal{I} generated by the homogeneous components of elements of E , we have $V_+(\mathcal{I}) = V_+(E)$; moreover, for any homogeneous ideal $\mathcal{I} \subseteq S$ we have

$$V_+(\mathcal{I}) = V_+\left(\bigcup_{q \geq n} (\mathcal{I} \cap S_q)\right)$$

for all n , by (2.1.8). Finally $V_+(\mathcal{I}) = V_+(\sqrt{+}\mathcal{I})$.

For $f \in S$, define $D_+(f) = D(f) \cap \text{Proj}(S) = \text{Proj}(S) \setminus V_+(f)$. We have $D_+(fg) = D_+(f) \cap D_+(g)$.

(2.3.4) *Proposition:* As f ranges over homogeneous elements of S_+ , the sets $D_+(f)$ form a base for the topology on $\text{Proj}(S)$.

[Proof: By (2.3.2.4), every $V_+(E)$ is an intersection of sets $V_+(f)$ for $f \in S_+$ homogeneous.]

(2.3.5) Let $f \in S_d$ ($d > 0$), and let \mathfrak{p} be a homogeneous prime ideal of S such that $f \notin \mathfrak{p}$. Then \mathfrak{p}_f is a prime ideal in S_f (0, 1.2.6), and $\mathfrak{p}_f \cap S_{(f)}$ is a prime ideal $\psi_f(\mathfrak{p}) \subseteq S_{(f)}$, consisting of elements x/f^n where $x \in \mathfrak{p} \cap S_{nd}$. This defines a map $\psi_f: D_+(f) \rightarrow \text{Spec}(S_{(f)})$. Given $g \in S_e$ ($e > 0$), we get a commutative diagram [see (2.2.2–3)]

$$\begin{array}{ccc} D_+(f) & \xrightarrow{\psi_f} & \text{Spec}(S_{(f)}) \\ \uparrow & & \uparrow \\ D_+(fg) & \xrightarrow{\psi_{fg}} & \text{Spec}(S_{(fg)}). \end{array}$$

(2.3.6) *Proposition* [Liu, 2.3.36(a)]: $\psi_f: D_+(f) \rightarrow \text{Spec}(S_{(f)})$ is a homeomorphism.

(2.3.7) *Corollary:* $D_+(f) = \emptyset$ iff f is nilpotent (still assuming $f \in S_+$ homogeneous).

(2.3.8) *Corollary:* Let $E \subseteq S_+$. The following are equivalent:

(a) $V_+(E) = X = \text{Proj}(S)$.

(b) Every element of E is nilpotent.

(c) The homogeneous components of every element of E are nilpotent.

(2.3.9) *Corollary:* If \mathcal{I} is a homogeneous ideal of S_+ , then $\sqrt{+}\mathcal{I}$ is the intersection of the homogeneous prime ideals of S_+ which contain \mathcal{I} .

[Proof: reducing to the case $\mathcal{I} = 0$, if $f \in S_+$ is not nilpotent we must find a homogeneous prime ideal not containing f . At least one homogeneous component of f is not nilpotent. Apply (2.3.7).]

(2.3.10) For any subset $Y \subseteq \text{Proj}(S)$, let $\mathfrak{i}_+(Y) = \{f \in S_+ : Y \subseteq V_+(f)\} = \mathfrak{i}(Y) \cap S_+$, an ideal of S_+ equal to its radical in S_+ .

(2.3.11) *Proposition:* (i) For every $E \subseteq S_+$, $\mathfrak{i}_+(V_+(E)) = \sqrt{+\mathcal{I}}$, where \mathcal{I} is the ideal generated by the homogeneous components of elements of E .

(ii) For every $Y \subseteq \text{Proj}(S)$, $V_+(\mathfrak{i}_+(Y))$ is the closure of Y in $\text{Proj}(S)$.

(2.3.12) *Corollary:* We have a containment-reversing bijection $Y \rightarrow \mathfrak{i}_+(Y)$, $\mathcal{I} \rightarrow V_+(\mathcal{I})$ between radical ideals of S_+ and closed subsets of $\text{Proj}(S)$. The union $Y_1 \cup Y_2$ corresponds to the intersection of ideals; the intersection of any family (Y_λ) corresponds to the radical of the sum of their ideals.

(2.3.13) *Corollary:* Let \mathcal{I} be a homogeneous ideal of S_+ . Then $V_+(\mathcal{I}) = \emptyset$ iff every element of S_+ has a power in \mathcal{I} .

(2.3.14) *Corollary:* Let (f_α) be homogeneous elements of S_+ . The open sets $D_+(f_\alpha)$ cover $\text{Proj}(S)$ iff every element of S_+ has a power in the ideal generated by the f_α 's.

(2.3.15) *Corollary:* With (f_α) as in (2.3.14), and $f \in S_+$, the following are equivalent: (a) $D_+(f) \subseteq \bigcup_\alpha D_+(f_\alpha)$; (b) $V_+(f) \supseteq \bigcap_\alpha V_+(f_\alpha)$; (c) some power of f is in the ideal generated by the f_α 's.

(2.3.16) *Corollary:* $\text{Proj}(S) = \emptyset$ iff every element of S_+ is nilpotent.

(2.3.17) *Corollary:* In the correspondence of (2.3.12), the irreducible closed subsets correspond to the prime ideals.

2.4 The scheme structure on $\text{Proj}(S)$.

(2.4.1) Let $f, g \in S_+$ be homogeneous. Transport the structure of affine scheme on $Y_f = \text{Spec}(S_{(f)})$ to $D_+(f)$ via the homeomorphism ψ_f in (2.3.6). Using (2.3.5), these structures on $D_+(f)$ and $D_+(g)$ agree canonically on $D_+(fg)$. This makes $\text{Proj}(S)$ a prescheme.

(2.4.2) *Proposition:* $\text{Proj}(S)$ is a *scheme* [i.e., it is separated].

The proof is by (I, 5.5.6), checking the second condition using (2.2.4).

(2.4.3) *Example:* For $S = K[t_1, t_2]$, $\text{Proj}(S)$ is the projective line from (I, 2.3.2).

(2.4.4) *Proposition:* Let S be a positively graded ring, $X = \text{Proj}(S)$.

(i) Let \mathfrak{N}_+ be the nilradical of S_+ . Then $X_{\text{red}} = \text{Proj}(S/\mathfrak{N}_+)$.

(ii) Suppose S essentially reduced. Then X is integral iff S is essentially integral.

(2.4.5) A *graded A -algebra* is a graded ring S with an A -algebra structure such that each S_n is an A -submodule; equivalently the structural homomorphism $A \rightarrow S$ has image in S_0 .

(2.4.6) *Proposition:* If S is a graded A -algebra, $X = \text{Proj}(S)$, then \mathcal{O}_X is a sheaf of A -algebras; that is, X is a scheme over $\text{Spec}(A)$.

(2.4.7) *Proposition:* Let S be a positively graded ring.

(i) For all $d > 0$, $\text{Proj}(S)$ is canonically isomorphic to $\text{Proj}(S^{(d)})$.

(ii) Let S' be the graded ring with $S'_0 = \mathbb{Z}$, $S'_n = S_n$ for all $n > 0$. Then $\text{Proj}(S) \cong \text{Proj}(S')$.

(2.4.8) *Corollary:* If S is a graded A -algebra, and S'_A is the graded A -algebra with $(S'_A)_0 = A$, $S'_n = S_n$ for $n > 0$, then $\text{Proj}(S) \cong \text{Proj}(S'_A)$.