Synopsis of material from EGA I
§9: Supplement on quasi-coherent sheaves, 9.5

9.5 Closed image of a prescheme; closure of a sub-prescheme [Liu, Ex. 3.17 (c–e)].

(9.5.1) *Proposition*: Let $f\colon X \to Y$ be a morphism of preschemes such that $f_*\mathcal{O}_X$ is quasi-coherent (as is the case if f is quasi-compact, and either f is separated or X is locally Noetherian). Then there is a smallest closed sub-prescheme $Y' \subseteq Y$ such that the inclusion $j\colon Y' \to Y$ majorizes f, or equivalently (4.4.1), such that the sub-prescheme $f^{-1}(Y') \subseteq X$ is equal to X.

(9.5.2) *Corollary*: More precisely, let $\mathcal{I}$ be the kernel of $\tilde{f}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Then the closed sub-prescheme Y' defined by $\mathcal{I}$ has the property in (9.5.1).

(9.5.3) A sub-prescheme Y' with the property in (9.5.1) is called the *closed image* of X under the morphism f.

[Remark: EGA seems to miss something here. Any sum of quasi-coherent ideal sheaves is quasi-coherent (4.1.1), so every ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ contains a unique largest quasi-coherent ideal sheaf $\mathcal{I}'$. For any $f\colon X \to Y$, let $\mathcal{I} = \ker \tilde{f}$. Then the closed subscheme Y' defined by $\mathcal{I}'$ is the closed image of X under f. One only needs $f_*\mathcal{O}_X$ quasi-coherent for (9.5.2) and some other results below, and for these it suffices that $\mathcal{I}$ is quasi-coherent, which holds, *e.g.*, whenever f is quasi-compact.]

(9.5.4) *Proposition*: If $f_*\mathcal{O}_X$ is quasi-coherent, then the underlying space of Y' is the closure of $f(X)$ in X.

(9.5.5) *Proposition* (transitivity of closed images): Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, let Y' be the closed image of X by f. Then the closed image of X by $g \circ f$ is equal to the closed image of Y' by the restriction $g'\colon Y' \to Z$ of g.

(9.5.6) *Corollary*: Let $f\colon X \to Y$ be an S-morphism such that $f_*\mathcal{O}_X$ is quasi-coherent and the closed image of X by f is equal to Y. Let Z be a scheme [*i.e.*, a separated prescheme] over S. If g_1, g_2 are morphisms $Y \to Z$ such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.

(9.5.7) In the case that X, Y are also separated, (9.5.6) says that f is an epimorphism in the category of separated preschemes over S.

(9.5.8) *Proposition*: Under the hypotheses of (9.5.1), if $V \subseteq Y$ is open, and f_V is the restricition of f to $f^{-1}(V)$, then $V \cap Y'$ is the closed image of $f^{-1}(V)$ by f_V.

(9.5.9) *Proposition*: Let Y' be the closed image of X under $f\colon X \to Y$.

(i) If X is reduced, then so is Y'.

(ii) If $f_*\mathcal{O}_X$ is quasi-coherent and X is irreducible (resp. integral), then so is Y'.

(9.5.10) *Proposition*: Let Y be a sub-prescheme of X such that the inclusion $i\colon Y \to X$ is quasi-compact. Then there is a smallest closed subscheme $\overline{Y}$ majorizing Y, its underlying space is the closure of Y, Y is open in its closure, and Y is equal to the restriction of $\overline{Y}$ to this open subset.

(9.5.11) *Corollary*: Under the hypotheses of (9.5.10), any section of $\mathcal{O}_{\overline{Y}}$ whose restriction to Y is zero, is itself zero.