Synopsis of material from EGA I

§9: Supplement on quasi-coherent sheaves, 9.1–2 and 9.6

9.1 Tensor product of quasi-coherent sheaves.

(9.1.1) *Proposition*: If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent (resp. coherent) sheaves on a prescheme (resp. locallly Noetherian prescheme) X, then so is $\mathcal{F} \otimes \mathcal{G}$, and it is of finite type if $\mathcal{F}$ and $\mathcal{G}$ are. If $\mathcal{F}$ is finitely presented and $\mathcal{G}$ is quasi-coherent (resp. coherent) then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent (resp. coherent).

(9.1.2) *Definition*: If X, Y are S-preschemes, and $\mathcal{F}$ (resp. $\mathcal{G}$) is an $\mathcal{O}_X$-module (resp. $\mathcal{O}_Y$-module), then $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$, or $\mathcal{F} \otimes_S \mathcal{G}$, denotes the tensor product $p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X \times_S Y}} p_2^*(\mathcal{G})$ on $X \times_S Y$.

Similar notation applies for products of n preschemes at a time. If $X = Y = S$, (9.1.2) reduces to the tensor product of $\mathcal{O}_S$-modules. We have canonically $p_1^*(\mathcal{F}) \cong \mathcal{F} \otimes_S \mathcal{O}_Y$ and similarly for p_2. $\mathcal{F} \otimes_S \mathcal{G}$ is a right-exact covariant functor in each variable.

(9.1.3) *Proposition*: If $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(C)$, $\mathcal{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$, then $\mathcal{F} \times_S \mathcal{G}$ is the sheaf associated to the $B \otimes_A C$-module $M \otimes_A N$.

(9.1.4) *Proposition*: Given S-morphisms $f \colon T \to X$, $g \colon T \to Y$, we have $(f, g)^*(\mathcal{F} \otimes_S \mathcal{G}) \cong f^*(\mathcal{F}) \otimes_{\mathcal{O}_T} g^*(\mathcal{G})$, canonically.

(9.1.5) *Corollary*: Given S-morphisms $f \colon X \to X'$, $g \colon Y \to Y'$, we have $(f \times_S g)^*(\mathcal{F}' \otimes_S \mathcal{G}') \cong f^*(\mathcal{F}') \otimes_S g^*(\mathcal{G}')$.

(9.1.6) *Corollary*: $\mathcal{F} \otimes_S \mathcal{G} \otimes_S \mathcal{H}$ and $(\mathcal{F} \otimes_S \mathcal{G}) \otimes_S \mathcal{H}$ are identified by the canonical isomorphism $X \times_S Y \times_S Z \cong (X \times_S Y) \times_S Z$.

(9.1.7) *Corollary*: $\mathcal{F} \otimes_S \mathcal{O}_S$ is identified with $\mathcal{F}$ by the canonical isomorphism $X \times_S S \cong X$.

(9.1.8) For $\mathcal{F}$ on X and a base change $X_{(S')} = X \times_S S'$, denote $\mathcal{F}_{(S')} = \mathcal{F} \otimes_S \mathcal{O}_{S'}$.

(9.1.9) *Proposition*: Given $S'' \underset{\phi'}{\to} S' \underset{\phi}{\to} S$, $(\mathcal{F}_{(\phi)})_{(\phi')} \cong \mathcal{F}_{(\phi \circ \phi')}$, canonically.

(9.1.10) *Proposition*: Let $f \colon X \to Y$ be an S-morphism, $\mathcal{G}$ an $\mathcal{O}_Y$-module. For the base change $S' \to S$, we have $(f_{(S')})^*(\mathcal{G}_{(S')}) = (f^*(\mathcal{G}))_{(S')}$.

(9.1.11) *Corollary*: $(\mathcal{F}_{(S')}) \otimes_{S'} (\mathcal{G}_{(S')})$ is identified with $(\mathcal{F} \otimes_S \mathcal{G})_{(S')}$ by the canonical isomorphism $(X_{(S')} \times_{S'} (Y_{(S')}) \cong (X \times_S Y)_{(S')}$.

(9.1.12) *Proposition*: With the notation of (9.1.2), let z be a point of $X \times_S Y$, $x = p_1(z)$, $y = p_2(z)$. The stalk $(\mathcal{F} \otimes_S \mathcal{G})_z$ is isomorphic to $(\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z) \otimes_{\mathcal{O}_z} (\mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z) = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z \otimes_{\mathcal{O}_y} \mathcal{G}_y$.

(9.1.13) *Corollary*: If $\mathcal{F}$, $\mathcal{G}$ are of finite type, then $\operatorname{Supp}(\mathcal{F} \otimes_S \mathcal{G}) = p_1^{-1}(\operatorname{Supp}(\mathcal{F})) \cap p_2^{-1}(\operatorname{Supp}(\mathcal{G}))$.

9.2 Direct image of a quasi-coherent sheaf.

(9.2.1) *Proposition*: Let $f \colon X \to Y$ be a morphism of preschemes. Suppose Y has an open affine covering (Y_α) such that each $f^{-1}(Y_\alpha)$ admits a finite affine covering (X_{α_i}), and each $X_{\alpha i} \cap X_{\alpha j}$ admits a finite affine covering. If $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then $f_* \mathcal{O}_X$ is quasi-coherent.

(9.2.2) *Corollary*: The conclusion of (9.2.1) holds under any of the conditions:
(a) f is separated and quasi-compact,
(b) f is separated and of finite type,
(c) f is quasi-compact and the underlying space of X is locally Noetherian.

9.6 Quasi-coherent sheaves of algebras; change of stucture sheaf.

(9.6.1) *Proposition*: Let X be a prescheme, $\mathcal{B}$ a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Then a $\mathcal{B}$-module $\mathcal{F}$ is quasi-coherent as a sheaf of modules on the ringed space $(X, \mathcal{B})$ iff $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module.

[The proof is by reduction to the case that X is affine. The proposition is a special property of preschemes, not valid for arbitrary ringed spaces.]

(9.6.2) A quasi-coherent $\mathcal{O}_X$-algebra is *of finite type* if every $x \in X$ has an affine neighborhood $U = \mathrm{Spec}(A)$ such that $B = \mathcal{B}(U)$ is a finitely-generated A-algebra. Then the same thing holds on $U_f = \mathrm{Spec}(A_f)$ for $f \in A$. It follows that if $\mathcal{B}$ is of finite type, then $\mathcal{B}|V$ is an $\mathcal{O}_V$-algebra of finite type for every open $V \subseteq X$.

(9.6.3) *Proposition*: If X is locally Noetherian, then every $\mathcal{O}_X$-algebra $\mathcal{B}$ of finite type is a coherent sheaf of rings.

(9.6.4) *Proposition*: Under the hypotheses of (9.6.3) a $\mathcal{B}$-module $\mathcal{F}$ is coherent iff $\mathcal{F}$ is of finite type as a $\mathcal{B}$-module and quasi-coherent as an $\mathcal{O}_X$-module. When this is so, if $\mathcal{G}$ is a sub-module or quotient of $\mathcal{F}$, then $\mathcal{G}$ is a coherent $\mathcal{B}$-module iff $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module.

(9.6.5) *Proposition*: Let X be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. Every quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ of finite type contains an $\mathcal{O}_X$-submodule of finite type which generates $\mathcal{B}$ as an $\mathcal{O}_X$-algebra.

(9.6.6) *Proposition*: Under the hypotheses of (9.6.5), every quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ is the inductive limit of its quasi-coherent sub-$\mathcal{O}_X$-algebras of finite type.