Synopsis of material from EGA I
§6: Finiteness conditions

6.1 Noetherian and locally Noetherian preschemes.

(6.1.1) *Definition*: X is *locally Noetherian* if it has a covering by open affines Spec(R) with R Noetherian. X is *Noetherian* if it has a finite such covering [Liu, 2.3.45].

If X is locally Noetherian, then $\mathcal{O}_X$ is coherent, a quasi-coherent sheaf of $\mathcal{O}_X$ modules is coherent iff it is locally finitely generated, and every quasi-coherent subsheaf of a coherent sheaf of $\mathcal{O}_X$ modules is coherent.

(6.1.2) *Proposition*: X is Noetherian iff it is locally Noetherian and quasi-compact; then its underlying space is a Noetherian topological space (but not conversely).

(6.1.3) *Proposition*: Spec(A) (a) is Noetherian iff (b) it is locally Noetherian iff (c) A is Noetherian [see Liu, Ex. 2.3.16].

(6.1.4) *Proposition*: Any open or closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian [Liu, 2.3.46].

(6.1.5) Since the tensor product of Noetherian algebras is not necessarily Noetherian, the product of two Noetherian schemes over a scheme S is not necessarily Noetherian.

(6.1.6) *Proposition*: If X is Noetherian, the nilradical $\mathcal{N}_X$ of $\mathcal{O}_X$ is nilpotent.

(6.1.7) *Corollary*: If X is Noetherian, then X is affine iff X_{red} is.

(6.1.8-9) A locally Noetherian topological space is locally connected, which implies that its connected components are open.

(6.1.10) *Proposition*: If X is a locally Noetherian topological space, the following are equivalent.
(a) The irreducible components of X are open.
(b) The irreducible components of X are the same as its connected components.
(c) The connected components of X are irreducible.
(d) Distinct irreducible components of X are disjoint.
If X is a prescheme, the above are also equivalent to:
(e) For every $x \in X$, Spec($\mathcal{O}_{X,x}$) is irreducible, that is, the nilradical of $\mathcal{O}_{X,x}$ is prime.

(6.1.11-12) *Corollary*: Let X be a locally Noetherian prescheme. Then X is irreducible iff X is connected, non-empty, and Spec($\mathcal{O}_{X,x}$) is irreducible for all $x \in X$. X is integral iff X is connected and $\mathcal{O}_{X,x}$ is an integral domain for all $x \in X$ [see Liu, last part of Ex. 4.4.4].

(6.1.13) *Proposition*: If X is a locally Noetherian prescheme, and $x \in X$ is such that the nilradical $\mathcal{N}_x$ of $\mathcal{O}_{X,x}$ is prime (resp. such that $\mathcal{O}_{X,x}$ is reduced; is a domain), then x has a neighborhood U which is irreducible (resp. reduced; integral) [Liu, Ex. 2.4.9].

6.2 Artinian preschemes.

(6.2.1) *Definition*: A prescheme is *Artinian* if it is affine and its ring is Artinian.

(6.2.2) *Proposition*: The following are equivalen for a prescheme X.
(a) X is Artinian.

(b) X is Noetherian and its underlying space is discrete.
(c) X is Noetherian and every point of X is closed (X is a T_1 space).
When the above hold, the underlying space of X is finite, and the ring A of X is the direct product of the (Artinian) local rings of the points of X.

6.3 Morphisms of finite type.
[What EGA takes as the definition (6.3.1) is the property in Liu, Prop. 3.2.2. Liu, Def. 3.2.1 is equivalent, using Liu 2.2.2 in one direction, and EGA (6.6.3) and (6.3.2-3) in the other.]

(6.3.1) *Definition*: A morphism $f\colon X \to Y$ is *of finite type* if Y can be covered by open affine subsets $V_\alpha = \mathrm{Spec}(A_\alpha)$ satisfying the property
(P): $f^{-1}(V_\alpha)$ is a finite union of affine opens $U_{\alpha,i} = \mathrm{Spec}(R_{\alpha,i})$ such that $R_{\alpha,i}$ is finitely generated as an algebra over A_α.

We also say that X is of finite type over Y, or a Y-prescheme of finite type.

(6.3.2) *Proposition*: If $f\colon X \to Y$ is of finite type, then property (P) holds for every affine open $W \subseteq Y$.

This means that the property that f is of finite type is *local on* Y.

(6.3.3) *Proposition*: $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is of finite type iff B is a finitely generated A-algebra ["if" is obvious, but "only if" requires proof].

(6.3.4) *Proposition* [Liu, 3.2.4]: (i) Any closed immersion is of finite type.
(ii) The composite of two morphisms of finite type is of finite type.
(iii) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are S-morphisms of finite type, then so is $f \times_S g$.
(iv) If $f\colon X \to Y$ is an S-morphism of finite type, then $f_{(S')}$ is of finite type for any base extension $S' \to S$.
(v) If $g \circ f$ is of finite type, and g is separated, then f is of finite type.
(vi) If f is of finite type, then so is f_{red}.

(6.3.5) *Corollary* [Liu, Ex. 3.2.2]: Let $f\colon X \to Y$ be an immersion. If the underlying space of Y is locally Noetherian, or if that of X is Noetherian, then f is of finite type.

(6.3.6) *Corollary*: Given $f\colon X \to Y$, $g\colon Y \to Z$. If $g \circ f$ is of finite type, and if X is Noetherian, or if $X \times_Z Y$ is locally Noetherian, then f is of finite type.

(6.3.7) *Proposition*: If X is of finite type over Y, and Y is (locally) Noetherian, then so is X.

(6.3.8) *Corollary*: If X is of finite type over S, then $X_{(S')}$ is (locally) Noetherian for every base extension $S' \to S$ such that S' is (locally) Noetherian.

(6.3.9) *Corollary*: If X is of finite type over a locally Noetherian prescheme S, then every S-morphism $f\colon X \to Y$ is of finite type.

[Remark: the preceding results hold for morphisms *locally* of finite type without the Noetherian hypotheses, which only serve to guarantee quasi-compactness; see §6.6.]

(6.3.10) *Proposition*: A morphism $f\colon X \to Y$ of finite type is surjective if and only if, for every algebraically closed field k, the map of k-points $\underline{X}(k) \to \underline{Y}(k)$ induced by f is surjective [*i.e.*, f is *geometrically surjective*].

6.4 Algebraic preschemes.

(6.4.1) Let K be a field. A prescheme X of finite type over K is called an *algebraic* K-prescheme, K the *ground field* of X [Liu, 2.3.47, Example 3.2.3].

An algebraic prescheme is automatically Noetherian.

(6.4.2) *Proposition*: Let X be an algebraic K-prescheme. A point $x \in X$ is closed iff $k(x)$ is a finite algebraic extension of K.

(6.4.3) *Corollary*: If $K = \overline{K}$ and X is an algebraic K-prescheme, then $\underline{X}(K) \to X_{\mathrm{cl}}$ is a bijection from the K-valued points of X to its closed points.

(6.4.4-6) *Proposition*: For an algebraic K-prescheme X, the following are equivalent.

(a) X is Artinian.

(b) The underlying space of X is discrete.

(c) The underlying space of X has finitely many closed points.

(c') The underlying space of X is finite.

(d) Every point of X is closed.

(e) $X = \mathrm{Spec}(A)$ where A is finite-dimensional as a K-vector space.

When the above hold, we say that X is *finite over* K, of *length* $l_K(X) \underset{def}{=} \dim_K(A)$. We have $l_K(X \sqcup Y) = l_K(X) + l_K(Y)$ and $l_K(X \times_K Y) = l_K(X) l_K(Y)$. If K' is a finite extension of K, then $X \otimes_K K'$ is finite over K', of length equal to $l_K(X)$.

(6.4.7-8) Let $[K : L]_s$ denote the *separable degree* of a finite field extension $L \subseteq K$, that is, the degree $[K' : L]$, where K' is the maximal separable algebraic extension of L inside K.

Corollary: Let X be finite over K and set $n = \sum_{x \in X} [k(x) : K]_s$. Then for every algebraically closed extension K' of K, the underlying space of $X \otimes_K K'$ has n points, identified bijectively with the K'-valued points of X.

We call n the *separable degree* or the *geometric number of points* of X over K. We have $n(X \sqcup Y) = n(X) + n(Y)$ and $n(X \times_K Y) = n(X) n(Y)$.

(6.4.9-10) *Proposition*: Let $f\colon X \to Y$ be a K-morphism of algebraic K-preschemes. Let K' be an algebraic closed extension of K, of infinite transcendence degree over K. Then f is surjective iff $\underline{X}(K') \to \underline{Y}(K')$ is surjective.

The infinite transcendence degree hypothesis can be removed (Volume IV).

(6.4.11) *Proposition*: If $f\colon X \to Y$ is of finite type, then for every $y \in Y$, the fiber $f^{-1}(y)$ is algebraic over $k(y)$, and for all closed points $x \in f^{-1}(y)$, $k(x)$ is a finite extension of $k(y)$.

(6.4.12) *Proposition*: Given morphisms $f\colon X \to Y$ and $g\colon Y' \to Y$, let $X' = X \times_Y Y'$ and $f' = f_{(Y')}\colon X' \to Y'$. Let $y' \in Y'$, $y = g(y')$. If the fiber $f^{-1}(y)$ is finite over $k(y)$, then so is $f'^{-1}(y')$ over $k(y')$, with the same degree and geometric number of points as $f^{-1}(y)$.

(6.4.13) One may interpret (6.4.11) as saying that morphisms of finite type describe "families of algebraic varieties" parametrized by the points of the target scheme Y.

6.5 Local determination of a morphism.

(6.5.1-3) *Proposition*: Let X, Y be S-preschemes, with Y of finite type over S. Let $x \in X$, $y \in Y$ lie over the same point $s \in S$.

(i) If $f, f'\colon X \to Y$ satisfy $f(x) = f'(x) = y$, and they induce the same (local) homomorphism of $\mathcal{O}_{S,s}$-algebras $f_x^\sharp = f_x'^\sharp$ from $\mathcal{O}_{Y,y}$ to $\mathcal{O}_{X,x}$, then f and f' coincide on a neighborhood of x.

(ii) [Liu, Ex. 3.2.4] Suppose further that S is locally Noetherian. Then every local $\mathcal{O}_{S,s}$-algebra homomorphism $\phi\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is induced by an S-morphism f from a neighborhood U of X to Y, such that $f(x) = y$.

In (ii) one can of course take $U = \mathrm{Spec}(B)$ affine, and such that $f(U) \subseteq W$, where $W = \mathrm{Spec}(A)$ is affine. If ϕ is injective, one can then take $f\colon U \to W$ to be induced by an injective ring homomorphism $A \to B$.

Also, in (ii), if X is of finite type over S, one can take f to be of finite type.

(6.5.4) *Proposition*: Let $f\colon X \to Y$ be a morphism of finite type, $x \in X$, $y = f(x)$.

(i) f is a local immersion at x (4.5.1) iff $f_x^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective.

(ii) Suppose further that Y is locally Noetherian. Then f is a local isomorphism at x iff $f_x^\sharp$ is an isomorphism.

(6.5.5) *Corollary*: Let $f\colon X \to Y$ be of finite type, X irreducible, x its generic point, and $y = f(x)$.

(i) f is a local immersion at some point of X iff $f_x^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective.

(ii) Suppose further that Y is irreducible and locally Noetherian. Then f is a local isomorphism at some point of X iff y is the generic point of Y (which by (0, 2.1.4) means f is *dominant*), and $f_x^\sharp$ is an isomorphism (that is, f is *birational* (2.2.9)).

6.6 Quasi-compact morphisms and morphisms locally of finite type.

(6.6.1) *Definition*[Liu, Ex. 2.3.17]: A morphism $f\colon X \to Y$ is *quasi-compact* if $f^{-1}(W)$ is quasi-compact for every quasi-compact open $W \subseteq Y$.

For f to be quasi-compact, it suffices that $f^{-1}(W)$ be quasi-compact for all W in a neighborhood base $\mathfrak{B}$ which consists of quasi-compact open sets. Note that the open affines of Y form such a base $\mathfrak{B}$. For example, if X is quasi-compact and Y is affine, then any morphism $f\colon X \to Y$ is quasi-compact.

If $Y = \bigcup_\alpha U_\alpha$ is an open covering, and the restriction of f to each $f^{-1}(U_\alpha)$ is quasi-compact, then so is f, that is, quasi-compactness of f is a local property on Y.

(6.6.2) *Definition*: $f\colon X \to Y$ is *locally of finite type* if for every $x \in X$ and $y = f(x)$ there are neighborhoods U of x and $V \supseteq f(U)$ of y such that f restricted to U is a morphism of finite type from U to V [note that we can always take $U = \mathrm{Spec}(B)$, $V = \mathrm{Spec}(A)$ affine here, and the condition is then that B is finitely generated as an A-algebra, by (6.3.3)].

(6.6.3) *Proposition*: f is of finite type iff f is quasi-compact and locally of finite type.

(6.6.4) *Proposition* [Liu, Ex. 2.3.17(a,b)]: (i) Any closed immersion is quasi-compact. An open immersion $X \hookrightarrow Y$ is quasi-compact if the underlying space of X is Noetherian, or if that of Y is locally Noetherian.

(ii) The composite of two quasi-compact morphisms is quasi-compact.

(iii) If $f\colon X \to Y$ is a quasi-compact S-morphism, so is $f_{(S')}$, for any base extension $S' \to S$.

(iv) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are quasi-compact S-morphisms, so is $f \times_S g$.

(v) If the composite $g \circ f$ of $f\colon X \to Y$ and $g\colon Y \to Z$ is quasi-compact, and if g is separated or the underlying space of X is Noetherian, then f is quasi-compact.

(vi) f is quasi-compact iff f_{red} is.

(6.6.5) *Proposition*: Let $f\colon X \to Y$ be quasi-compact. Then f is dominant iff for each generic point y of an irreducible component of Y, $f^{-1}(y)$ contains the generic point of an irreducible component of X.

(6.6.6) *Proposition*: (i) Every local immersion is locally of finite type.

(ii) The composite of two morphisms locally of finite type is again so.

(iii) If $f\colon X \to Y$ is an S-morphism locally of finite type, so is $f_{(S')}$, for any base extension $S' \to S$.

(iv) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are S-morphisms locally of finite type, so is $f \times_S g$.

(v) If $g \circ f$ is locally of finite type, then so is f.

(vi) If f is locally of finite type, so is f_{red}.

(6.6.7) *Corollary*: Let X, Y be S-preschemes locally of finite type. If S is locally Noetherian, then so is $X \times_S Y$.

(6.6.8) Remark: (6.3.10) still holds if f is locally of finite type. Similarly, (6.4.2) and (6.4.9) hold if X, Y are only assumed locally of finite type over K.