

SYNOPSIS OF MATERIAL FROM EGA I
§3: PRODUCTS

3.1 Disjoint union of preschemes.

A disjoint union of ringed spaces $X = \bigsqcup_{\alpha} X_{\alpha}$ (defined in the obvious way, with each X_{α} open and closed in X) is a prescheme if each X_{α} is. To give a morphism $X \rightarrow Y$ it is equivalent to give morphisms $X_{\alpha} \rightarrow Y$ for each α . In particular, if each X_{α} is an S -prescheme, then so is X . If $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ are affine, then $X \sqcup Y = \text{Spec}(A \otimes B)$ is affine [this does not hold for infinite disjoint unions].

3.2 Product of preschemes.

(3.2.1) *Definition* [Liu, 3.1.1]: Given S -preschemes X, Y , we say that an S -prescheme Z , together with S -morphisms $p_1: Z \rightarrow X, p_2: Z \rightarrow Y$, is their *product*, if for every S -prescheme T , the correspondence $f \mapsto (p_1 \circ f, p_2 \circ f)$ is a bijection

$$\text{Hom}_S(T, Z) \cong \text{Hom}_S(T, X) \otimes \text{Hom}_S(T, Y).$$

In other words, the definition says that Z is the product of X and Y in the category of S -preschemes. In particular, the product is unique up to a canonical isomorphism. We usually denote it by $X \times_S Y$, suppressing the morphisms p_1, p_2 (called the *canonical projections*) from the notation. Given S -morphisms $g: T \rightarrow X$ and $h: T \rightarrow Y$, we write (g, h) for the corresponding morphism $f: T \rightarrow X \times_S Y$. Given $u: X \rightarrow X'$ and $v: Y \rightarrow Y'$, we write $u \times v$ for the induced morphism $X \times_S Y \rightarrow X' \times_S Y'$.

When $S = \text{Spec}(A)$, we often write A in place of S in the notation.

(3.2.2) *Proposition*: $\text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(C) = \text{Spec}(B \otimes_A C)$.

This follows from (2.2.4) and the universal property of the tensor product [Liu, 2.3.23 and 1.1.14].

(3.2.3) *Corollary*: In the setting of (3.2.2), if $T = \text{Spec}(D)$ and $g: T \rightarrow \text{Spec}(B), h: T \rightarrow \text{Spec}(C)$ correspond to A -algebra homomorphisms $\rho: B \rightarrow D, \sigma: C \rightarrow D$, then $(g, h): T \rightarrow \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(C)$ corresponds to the A -algebra homomorphism $\tau: B \otimes_A C \rightarrow D$ such that $\tau(b \otimes c) = \rho(b)\sigma(c)$.

(3.2.4) *Proposition*: If $f: S' \rightarrow S$ is a monomorphism [this means that composition with f is an injection $\text{Hom}(T, S') \rightarrow \text{Hom}(T, S)$ for every T], and X, Y are S' -schemes, considered also as S' -schemes via f , then a product $X \times_S Y$ is also a product $X \times_{S'} Y$, and conversely.

(3.2.5) *Corollary*: If $S' \subseteq S$ is open, and $f: X \rightarrow S, g: Y \rightarrow S$ have images contained in S' , then a product of X and Y as S -schemes is a product as S' -schemes, and conversely.

(3.2.6) *Theorem* [Liu, 1.1.2]: The product of S -schemes $X \times_S Y$ always exists.

The proof proceeds in steps [Liu gives a different stepwise proof].

(3.2.6.1) *Lemma* [Liu, 3.1.4 (d)]: If $Z = X \times_S Y$ is a product and $U \subseteq X, V \subseteq Y$ are open, then $W = p_1^{-1}(U) \cap p_2^{-1}(V)$ is the product $U \times_S V$.

(3.2.6.2) *Lemma*: Given S -morphisms $p_1: Z \rightarrow X, p_2: Z \rightarrow Y$, and open coverings $X = \bigcup_{\alpha} U_{\alpha}, Y = \bigcup_{\beta} V_{\beta}$, if $p_1^{-1}(U_{\alpha}) \cap p_2^{-1}(V_{\beta})$ is a product $U_{\alpha} \times_S V_{\beta}$ for all α, β , then $Z = X \times_S Y$.

(3.2.6.3) Suppose we are only given X, Y and the open coverings, and we assume that the product $U_\alpha \times_S V_\beta$ exists for all α, β . By (3.2.6.1), each $(U_{\alpha_1} \cap U_{\alpha_2}) \times_S (V_{\beta_1} \cap V_{\beta_2})$ exists and is identified with an open subscheme of each $U_{\alpha_i} \times_S V_{\beta_i}$, $i = 1, 2$. We can glue the schemes $U_\alpha \times_S V_\beta$ along these open subschemes to construct a scheme Z , which is then the product $X \times_S Y$ by (3.2.6.2).

(3.2.6.4) Let X, Y be S -schemes, with structure morphisms $\phi: X \rightarrow S, \psi: Y \rightarrow S$. If $S = \bigcup_i S_i$ is an open covering, set $X_i = \phi^{-1}(S_i), Y_i = \psi^{-1}(S_i)$. If the products $X_i \times_S Y_i$ exist, then we can construct $X \times_S Y$ by gluing them. By (3.2.5), it is equivalent that the products $X_i \times_{S_i} Y_i$ exist.

(3.2.6.5) Given any $\phi: X \rightarrow S, \psi: Y \rightarrow S$, we can cover S by affines S_i , then cover each X_i, Y_i by affines $U_{i,\alpha}, V_{i,\beta}$. So Theorem (3.2.6) is reduced to the affine case, (3.2.2).

(3.2.7) *Corollary:* If $Z = X \times_S Y, S' \subseteq S$ is open, and $U \subseteq X, V \subseteq Y$ are open subsets lying over S' , then $p_1^{-1}(U) \circ p_2^{-1}(V) = U \times_{S'} V$. If $f: T \rightarrow X, g: T \rightarrow Y$ have images contained in U, V respectively, then $(f, g)_{S'}$ is the restriction of (f, g) to $U \times_{S'} V$.

(3.2.8) The product of two disjoint unions $X = \bigsqcup X_\alpha, Y = \bigsqcup_\beta Y_\beta$ of S -schemes is the disjoint union of the products $X_\alpha \times_S Y_\beta$.

3.3 Formal properties of the product; change of base.

(3.3.1) Everything in this section except (3.3.13) and (3.3.15) is valid in any category in which products of objects X, Y over an object S exist.

(3.3.2) $X \times_S Y$ is a covariant bifunctor of X and Y [Liu, 1.1.4 (c)].

(3.3.3) *Proposition:* The projection p_1 (resp. p_2) is a functorial isomorphism of $X \times_S S$ (resp. $S \times_S X$) on X , with inverse $(1_X, \phi)$, where $\phi: X \rightarrow S$ is the structure morphism.

In other words, up to canonical isomorphism, we have $X \times_S S = S \times_S X = X$.

(3.3.4) *Corollary:* Under the identifications $X = X \times_S S, Y = S \times_S Y$, the projections $X \times_S Y \rightarrow X, X \times_S Y \rightarrow Y$ are identified with $(1_X, \phi_Y), (\phi_X, 1_Y)$.

(3.3.5) One can define the product of n S -preschemes in the obvious way; it exists and is associative, e.g., $X_1 \times X_2 \times X_3 = (X_1 \times X_2) \times X_3 = X_1 \times (X_2 \times X_3)$, up to canonical natural isomorphisms.

(3.3.6-8) Suppose given $\phi: S' \rightarrow S$, making S' an S -prescheme. For any S -prescheme X , the product $X \times_S S'$ is an S' -prescheme with structure morphism the second projection. We denote $X \times_S S'$, regarded as a scheme over S' , by $X_{(S')}$, or by $X_{(\phi)}$. One says that $X_{(S')}$ is obtained by *base change from S to S'* [Liu, 3.1.7].

Given an S -morphism $f: X \rightarrow Y$, we write $f_{(S')}: X_{(S')} \rightarrow Y_{(S')}$ for $f \times_S 1_{S'}$. Then base change is a covariant functor from S -preschemes to S' -preschemes.

Base change is right adjoint to the “forgetful” functor which regards every S' -prescheme as an S -prescheme via ϕ . That is, S' -morphisms $T \rightarrow X_{(S')}$ are in natural bijection with S -morphisms $T \rightarrow X$ when T is regarded as an S -scheme.

(3.3.9) *Proposition* (“transitivity of base change”): Given $S'' \rightarrow S' \rightarrow S$, we have a canonical natural isomorphism $X_{(S'')} = (X_{(S')})_{(S'')}$.

(3.3.10) *Corollary:* There is a canonical natural isomorphism

$$(X \times_S Y)_{(S')} = X_{(S')} \times_{S'} Y_{(S')}.$$

(3.3.11) *Corollary:* If Y is an S -prescheme and $f: X \rightarrow Y$ is a morphism making X a Y -prescheme, and hence an S -prescheme, then there is a canonical natural isomorphism $X_{(S')} = X \times_Y Y_{(S')}$, and $f_{(S')}$ is identified with the projection $X \times_Y Y_{(S')} \rightarrow Y_{(S')}$.

(3.3.12) If f and g are monomorphisms, so is $f \times_S g$. In particular, so is $f_{(S')}$ for any base change.

(3.3.13) If $S = \text{Spec}(A)$, $S' = \text{Spec}(A')$ are affine, with $S' \rightarrow S$ given by a ring homomorphism $A \rightarrow A'$, making A' an A -algebra, then given an S -prescheme X , we also write $X_{(A')}$ or $X \otimes_A A'$ for $X_{(S')}$, since when $X = \text{Spec}(B)$, we have $X_{(S')} = \text{Spec}(B \otimes_A A')$, the affine scheme associated to the extension of scalars to A' of the A -algebra B .

(3.3.14) With the notation of (3.3.6), for every S -morphism $f: S' \rightarrow X$, $f' = (f, 1_{S'})_S$ is an S' -morphism from S' to $X_{(S')}$, that is, a section of $X_{(S')}$ over S' . Conversely, given any such section, its composite with the projection $X_{(S')} \rightarrow X$ is an S -morphism $S' \rightarrow X$. Thus we have a canonical bijection

$$\text{Hom}_S(S', X') \cong \text{Hom}_{S'}(S', X_{(S')}).$$

The morphism f' is called the *morphism graph* of f and denoted Γ_f .

(3.3.15) As every prescheme X is uniquely a prescheme over \mathbb{Z} , (3.3.14) implies that the X -sections of $X \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ correspond bijectively to morphisms $X \rightarrow \text{Spec}(\mathbb{Z}[t])$, thus to ring homomorphisms $\mathbb{Z}[t] \rightarrow \mathcal{O}_X(X)$, and thus to global sections of \mathcal{O}_X .

3.4 Points of a prescheme with values in a prescheme; geometric points.

(3.4.1) Denote by $X(T)$ the set $\text{Hom}(T, X)$ of morphisms $T \rightarrow X$ of preschemes. Its elements are called *points of X with values in T* . For fixed X , $T \rightarrow X(T)$ is a contravariant functor in T , from preschemes to sets. A morphism $g: X \rightarrow Y$ induces a natural transformation of functors $X(T) \rightarrow Y(T)$.

(3.4.2) Given sets and maps $\phi: P \rightarrow R$, $\psi: Q \rightarrow R$, the subset $\{(p, q) \in P \times Q : \phi(p) = \psi(q)\}$ is called the *fiber product* of P and Q over R (relative to the given maps). We can interpret the definition (3.2.1) of the product of S -preschemes as the identity

$$(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T).$$

(3.4.3) If we fix S and consider only S -morphisms of S -preschemes, we write $X(T)_S$ for $\text{Hom}_S(T, X)$, suppressing the S from the notation when it will not cause confusion. Its elements are the *points* (or *S -points*) of the S -prescheme X with value in the S -prescheme T . In particular, the S -sections of X are just the points of X with value in S . The formula in (3.4.2) may also be written

$$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S.$$

More generally if Z is an S -prescheme and X, Y, T are Z -preschemes (hence *ipso facto* S -preschemes), we have

$$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S.$$

To show that an S -prescheme W with S -morphisms $p_1: W \rightarrow X$, $p_2: W \rightarrow Y$ is a product of X and Y over an S -prescheme Z , it suffices to show that for every S -prescheme T , the diagram

$$\begin{array}{ccc} W(T)_S & \longrightarrow & X(T)_S \\ \downarrow & & \downarrow \\ Y(T)_S & \longrightarrow & Z(T)_S \end{array}$$

is a fiber product of sets.

(3.4.4) In the preceding, when $T = \text{Spec}(B)$ or $S = \text{Spec}(A)$, we often replace T and/or S by B , A in the notation, and refer to the elements of $X(B)$, $X(B)_A$ as *points of X with values in B* , or *points of the A -prescheme X with values in the A -algebra B* . Note that $X(B)$, $X(B)_A$ are *covariant* functors from rings (resp. A -algebras) to sets.

(3.4.5) As a special case, if $T = \text{Spec}(A)$ where A is a local ring, then the points in $X(A)$ correspond bijectively to pairs consisting of a point $x \in X$ and a ring homomorphism $\mathcal{O}_{X,x} \rightarrow A$; see (2.4.4). The point x is called the *location* of the corresponding point in $X(A)$.

Still more specially, points of X with values in a field K are called *geometric points*. A geometric point corresponds to a point $x \in X$ and a field extension $k(x) \hookrightarrow K$. One speaks of a geometric point *located at x* , with *value field K* . There is a map $X(K) \rightarrow X$ sending a geometric point to its location.

If $\text{Spec}(K)$ is an S -prescheme, that is, K is an extension of $k(s)$ for some $s \in S$, and X is an S -prescheme, elements of $X(K)_S$ are *geometric points of X lying over s with values in K* . Such a point is given by its location $x \in X$, a point lying over $s \in S$, and a $k(s)$ -homomorphism $k(x) \rightarrow K$; note that the structure morphism of X as an S -scheme makes $k(x)$ an extension of $k(s)$.

In particular, if $S = \text{Spec}(K) = \{\xi\}$, then $X(K)_K$ is identified with the set of points of x such that $k(x) = K$, called *K -rational points of the K -prescheme X* . If K' is an extension of K , then $X(K')_K$ corresponds bijectively with the set of K' -rational points of $X_{(K')}$ (3.3.14).

(3.4.6) *Lemma:* Let X_1, \dots, X_n be S -preschemes, $s \in S$, $x_i \in X_i$ lying over s . There exists an extension K of $k(s)$ and a geometric point of the product $Y = X_1 \times_S \cdots \times_S X_n$ with values in K , whose projection on each X_i is located at x_i .

(3.4.7) *Proposition:* Given S -preschemes X_1, \dots, X_n and points $x_i \in X_i$, there exists a point y of the product $Y = X_1 \times_S \cdots \times_S X_n$ whose projection on each X_i is x_i , if and only if all the points x_i lie over the same point $s \in S$.

Denoting the underlying set of X by (X) , the proposition says that $(X \times_S Y)$ maps *surjectively* on $(X) \times_{(S)} (Y)$. This map is *not injective* in general—more than one point of $X \times_S Y$ can have the same projections $x \in X$ and $y \in Y$. Example: let X, Y, S be the spectra of fields K, K', k . The algebra $K \otimes_k K'$ need not be a field, and can have more than one prime ideal—see (3.4.9).

(3.4.8) *Corollary:* Let $f_{(S')}: X_{(S')} \rightarrow Y_{(S')}$ be obtained from a morphism $f: X \rightarrow Y$ of S -preschemes by base change. Let $p: X_{(S')} \rightarrow X$, $q: Y_{(S')} \rightarrow Y$ be the canonical projections. Then for every subset $M \subseteq X$, we have $q^{-1}(f(M)) = f_{(S')}(p^{-1}(M))$.

(3.4.9) *Proposition:* Let X, Y be S -preschemes, $x \in X, y \in Y$ lying over the same point $s \in S$. The points in $X \times_S Y$ which project on x and y correspond bijectively to isomorphism classes of extensions of $k(s)$ generated by $k(x)$ and $k(y)$, or equivalently, to the points of $\text{Spec}(k(x) \otimes_{k(s)} k(y))$.

3.5 Surjections and injections.

(3.5.1) Consider the following two propositions about a property \mathbf{P} of morphisms of preschemes.

(i) If f, g are S -morphisms with property \mathbf{P} , then $f \times_S g$ has property \mathbf{P} .

(ii) If f is an S -morphism with property \mathbf{P} , then every base change $f_{(S')}$ has property \mathbf{P} .

We always have (i) implies (ii). If the composite of two morphisms with property \mathbf{P} has property \mathbf{P} , then (ii) implies (i). [When (ii) holds, \mathbf{P} is called *stable under base change*—Liu, 3.1.23.]

(3.5.2) *Proposition:* (i) If $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ are surjective S -morphisms, then $f \times_S g$ is surjective.

(ii) If $f: X \rightarrow Y$ is a surjective S -morphism, then so is any base change $f_{(S')}$ [Liu, Ex. 3.1.8].

In fact, (3.4.8) with $M = X$ gives (ii), and then (3.5.1) gives (i).

(3.5.3) *Proposition:* A morphism $f: X \rightarrow Y$ is surjective if and only if, for every field K and morphism $\text{Spec}(K) \rightarrow Y$, there exists an extension K' of K and a morphism $\text{Spec}(K') \rightarrow X$ making the diagram

$$\begin{array}{ccc} \text{Spec}(K') & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \text{Spec}(K) & \longrightarrow & Y \end{array}$$

commute. In other words, f is surjective iff *every geometric point of Y with values in K is the image of a geometric point of X with values in some extension of K* .

(3.5.4) *Definition:* A morphism $f: X \rightarrow Y$ is *universally injective* if $X(K) \rightarrow Y(K)$ is injective for every field K .

In particular, a monomorphism of preschemes is universally injective.

(3.5.5) For f to be universally injective, it suffices that the condition hold for all algebraically closed K .

(3.5.6) *Proposition:* (i) The composite of two universally injective morphisms is universally injective.

(ii) If $g \circ f$ is universally injective, then so is f .

(3.5.7) *Proposition:* (i) If f, g are universally injective S -morphisms, then so is $f \times_S g$.

(ii) If f is universally injective, then so is any base change $f_{(S')}$.

(3.5.8) *Proposition:* $f: X \rightarrow Y$ is universally injective if and only if f is injective, and for every $x \in X$, the monomorphism $f_x^\sharp: k(f(x)) \rightarrow k(x)$ makes $k(x)$ a purely inseparable algebraic extension of $k(f(x))$ [$K \subseteq L$ is *purely inseparable* if every element of $L \setminus K$ is

inseparable over K ; equivalently, if L has a unique embedding into any algebraically closed extension of K ; in characteristic zero this implies $K = L$].

(3.5.9) *Corollary*: The canonical morphism $\mathrm{Spec}(S^{-1}A) \rightarrow \mathrm{Spec}(A)$ is universally injective.

(Actually, such a morphism is a monomorphism.)

(3.5.10) *Corollary*: Let $f: X \rightarrow Y$ be universally injective, $g: Y' \rightarrow Y$ any morphism, and put $X' = X_{(Y')} = X \times_Y Y'$. Then the universally injective morphism $f_{(Y')}$ is a bijection of the underlying set of X' onto $g^{-1}(f(X))$. Moreover, for any field K , $X'(K)$ is identified with the subset of $Y'(K)$ which is the preimage of $X(K) \subseteq Y(K)$ via the map $Y'(K) \rightarrow Y(K)$ induced by g .

(3.5.11) A morphism $f = (\psi, \phi): X \rightarrow Y$ is said to be *injective* if ψ is injective. Then f is universally injective if and only if every base change $f_{(Y')}: X_{(Y')} \rightarrow Y'$ is injective; this explains the terminology.

3.6 Fibers.

(3.6.1) *Proposition* [Liu, 3.1.16]: Suppose given a morphism $f: X \rightarrow Y$, a point $y \in Y$, and an ideal \mathfrak{a}_y in \mathcal{O}_y such that $\sqrt{\mathfrak{a}_y} = \mathfrak{m}_y$. Then the projection $p: X \times_Y \mathrm{Spec}(\mathcal{O}_y/\mathfrak{a}) \rightarrow X$ is a homeomorphism onto the fiber $f^{-1}(y)$, considered as a subspace of X .

(3.6.2) In the rest of the text, when we consider a fiber $f^{-1}(y)$ to be a prescheme, we are referring to the prescheme over $k(y)$ obtained by transporting the prescheme structure on $X \times_Y \mathrm{Spec}(k(y))$ onto $f^{-1}(y)$ via the homeomorphism given by the projection to X [Liu, 3.1.17].

We also write the above product as $X \otimes_Y k(y)$ or $X \otimes_{\mathcal{O}_y} k(y)$. More generally, if B is an \mathcal{O}_y -algebra, we denote $X \times_Y \mathrm{Spec}(B)$ by $X \otimes_Y B$ or $X \otimes_{\mathcal{O}_y} B$.

With these conventions, it follows from (3.5.10) that the points of X with values in an extension K of $k(y)$ are identified with the points of $f^{-1}(y)$ with values in K .

(3.6.3) Given $f: X \rightarrow Y$, $g: Y \rightarrow Z$, $h = g \circ f$, the fiber $h^{-1}(z)$ is isomorphic to

$$X \times_Z \mathrm{Spec}(k(z)) = X \times_Y g^{-1}(z).$$

In particular, if $U \subseteq X$ is open, then $U \cap f^{-1}(y)$, considered as an open sub-prescheme of $f^{-1}(y)$, is isomorphic to the fiber $(f_U)^{-1}(y)$, where f_U is the restriction of f to U .

(3.6.4) *Proposition* (“transitivity of fibers”): Given $f: X \rightarrow Y$, $g: Y' \rightarrow Y$, let $X' = X_{(Y')}$ and $f' = f_{(Y')}$. For every $y' \in Y'$, setting $y = g(y')$, the prescheme $f'^{-1}(y')$ is isomorphic to $f^{-1}(y) \times_{k(y)} k(y')$.

In particular, if V is an open neighborhood of y and f_V denotes the restriction of f to $f^{-1}(V)$, then the preschemes $f^{-1}(y)$ and $f_V^{-1}(y)$ are canonically isomorphic.

(3.6.5) *Proposition*: Let $f: X \rightarrow Y$ be a morphism, y a point of Y , $Z = \mathrm{Spec}(\mathcal{O}_{Y,y})$, $p: X \times_Y Z \rightarrow X$ the projection. Then p is a homeomorphism of $X \times_Y Z$ onto the subspace $f^{-1}(Z)$ of X , where Z is identified with a subspace of Y as in (2.4.2), and for every $t \in X \times_Y Z$, setting $z = p(t)$, the induced homomorphism $\tilde{p}_t^\#: \mathcal{O}_z \rightarrow \mathcal{O}_t$ is an isomorphism.

3.7 Application: reduction of a prescheme mod \mathcal{I} (not used later; intended for readers familiar with classical algebraic geometry).

(3.7.1) If X is an A -prescheme, and $\mathcal{I} \subseteq A$ is an ideal, then $X_0 = X \otimes_A (A/\mathcal{I})$ is an (A/\mathcal{I}) -prescheme, said to be obtained from X by *reduction mod \mathcal{I}* .

(3.7.2) This terminology is primarily used when A is local, \mathcal{I} is maximal, and then X_0 is a prescheme over the residue field $k = A/\mathcal{I}$.

If A is an integral domain with fraction field K , we also have the K -prescheme $X' = X \otimes_A K$, and by abuse of language we also say that X_0 is obtained from X' by reduction mod \mathcal{I} . As a typical example, A might be the ring of p -adic integers $\mathbb{Z}_{(p)}$, so that X' is a prescheme over \mathbb{Q} , while X_0 is a prescheme over the finite field $\mathbb{Z}/p\mathbb{Z}$. In this case, $Y = \text{Spec}(A)$ has two points, the closed point $y = \mathcal{I}$, and the generic point (0) , which is the unique point of an open set $U = \text{Spec}(K)$ of Y . Then X' is just the open prescheme $\phi^{-1}(U) \subseteq X$ for the structure morphism $X \rightarrow Y$. Suppose all this takes place inside some fixed ambient Noetherian A -prescheme $\psi: P \rightarrow Y$, for example, projective space \mathbb{P}_A^n , and let $P' = \psi^{-1}(U)$. Given a closed subscheme $X' \subseteq P'$, there is a smallest closed subscheme $X = \overline{X'} \subseteq P$ containing X' , allowing us to regard X' as canonically of the form $X \otimes_A K$. Then its reduction mod \mathcal{I} , $X_0 = X \otimes_A k$, is just the fiber of X over the closed point. [Similarly, given a projective scheme $X' \subseteq \mathbb{P}_{\mathbb{Q}}^n$ defined over \mathbb{Q} , we can regard $\text{Spec}(\mathbb{Q})$ as the generic point of $\text{Spec}(\mathbb{Z})$, and obtain the reduction mod (p) of X' as the fiber of the closure X of X' in $\mathbb{P}_{\mathbb{Z}}^n$, over the closed point $(p) \subseteq \mathbb{Z}$.]

(3.7.3) A source of some confusion in this situation has been the fact that if A is a discrete valuation ring and X is *proper* over A (see volume II), for instance if X is projective, then the points of X with values in A and the points with values in K are in canonical bijection. This sometimes leads results to be stated for X' , which are really results about X , and which if stated in the latter form would be valid without the assumption that the ground ring A is local of dimension 1.