Synopsis of material from EGA Ch. I
§1: Affine schemes, 1.4–1.6

1.4 Quasi-coherent sheaves on a prime spectrum.

(1.4.1) *Theorem:* Let $X = \mathrm{Spec}(A)$, $V \subseteq X$ open and quasi-compact, and $\mathcal{F}$ a sheaf of $(\mathcal{O}_X|V)$-modules. The following are equivalent:

(a) There is an A-module M such that $\mathcal{F} \cong \widetilde{M}|V$.

(b) V has an open covering by subsets of the form $V_i = D(f_i)$ such that $\mathcal{F}|V_i \cong \widetilde{M_i}$ for some A_{f_i}-module M_i, for each i.

(c) $\mathcal{F}$ is quasi-coherent (0, 5.1.3).

(d) For every $f \in A$ such that $D(f) \subseteq V$, the following two conditions hold:

(d 1) for every section $s \in \mathcal{F}(D(f))$, there is some n such that $f^n s$ extends to a section of $\mathcal{F}$ on V;

(d 2) for every section $t \in \mathcal{F}(V)$ such that $t|_{D(f)} = 0$, there is some n such that $f^n t = 0$.

One also has the variant of (d) in which instead of assuming $D(f) \subseteq V$, we allow any $D(f)$ and replace $D(f)$ with $D(f) \cap V$ in (d 1-2).

(1.4.2) *Corollary:* Every quasi-coherent sheaf on a quasi-compact open subset of X is the restriction of a quasi-coherent sheaf on X.

(1.4.3) *Corollary:* Every quasi-coherent $\mathcal{O}_X$-algebra is isomorphic to $\widetilde{B}$ for an A-algebra B, and every quasi-coherent $\widetilde{B}$-module is isomorphic to $\widetilde{N}$ for some B-module N.

1.5 Coherent sheaves on a prime spectrum.

(1.5.1) *Theorem:* Let $X = \mathrm{Spec}(A)$, where A is *Noetherian*. Let $V \subseteq \mathrm{Spec}(A)$ be open and $\mathcal{F}$ a sheaf of $(\mathcal{O}_X|V)$-modules. The following are equivalent:

(a) $\mathcal{F}$ is coherent.

(b) $\mathcal{F}$ is quasi-coherent and finitely generated.

(c) $\mathcal{F} \cong \widetilde{M}|V$ for a finitely generated A-module M.

(1.5.2) *Corollary:* Under the hypotheses of (1.5.1), $\mathcal{O}_X$ is a coherent sheaf of rings.

(1.5.3) *Corollary:* Under the hypotheses of (1.5.1), every coherent sheaf on an open subset of X is the restriction of a coherent sheaf on X.

(1.5.4) *Corollary:* Under the hypotheses of (1.5.1), every quasi-coherent sheaf on X is the direct limit of coherent sub-sheaves.

1.6 Functorial properties of quasi-coherent sheaves on prime spectra.

(1.6.1) Let $\phi\colon A' \to A$ be a ring homomorphism and

$$^a\phi\colon X = \mathrm{Spec}(A) \to X' = \mathrm{Spec}(A')$$

the associated continuous map (1.2.1). We define a canonical homomorphism of sheaves of rings

$$\widetilde{\phi}\colon \mathcal{O}_{X'} \to {}^a\phi_*\mathcal{O}_X$$

as follows. Given $f' \in A$, let $f = \phi(f')$. Then ${}^a\phi^{-1}(D(f')) = D(f)$, $\Gamma(D(f'), \widetilde{A'}) = A'_{f'}$, and $\Gamma(D(f), \widetilde{A}) = A_f$. The canonical homomorphism $\phi_{f'} \colon A'_{f'} \to A_f$ is thus identified with a ring homomorphism

$$\Gamma(D(f'), \widetilde{A'}) \to \Gamma(D(f'), {}^a\phi_* \widetilde{A}).$$

These homomorphisms are compatible with restriction, and the sets $D(f')$ form a base of open sets, so this defines $\widetilde{\phi}$. Then $({}^a\phi, \widetilde{\phi})$ is a morphism of ringed spaces (0, 4.1.1) [Liu, 2.3.14]

$$\Phi \colon (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'}).$$

Note that the stalk homomorphism $\widetilde{\phi}^\sharp_x$ (0, 3.7.1) is just the canonical homomorphism $\phi_x \colon A'_{x'} \to A_x$, where $x' = {}^a\phi(x)$ (0, 1.5.1).

(1.6.2) *Example:* [Liu, 2.3.15] Consider $\phi \colon A \to S^{-1}A$. In (1.2.6), we saw that ${}^a\phi$ is a homeomorphism of $Y = \operatorname{Spec}(S^{-1}A)$ onto the subspace of points $x \in X = \operatorname{Spec}(A)$ such that $\mathfrak{i}_x \cap S = \emptyset$. In this case, for $x = {}^a\phi(y)$, the stalk homorphism $\widetilde{\phi}^\sharp_y \colon \mathcal{O}_x \to \mathcal{O}_y$ is an isomorphism (0, 1.2.6), which is to say, $\mathcal{O}_Y$ is the restriction of $\mathcal{O}_X$ to the subspace Y.

(1.6.3) *Proposition:* Let M be an A-module, and denote it by $M_{[\phi]}$ when regarded as an A'-module. Then there is a natural isomorphism of functors $\Phi_*(\widetilde{M}) \cong (M_{[\phi]})^\sim$. If $M = B$ is an A-algebra, the isomorphism is an isomorphism of sheaves of $\mathcal{O}_{X'}$-algebras.

(1.6.4) *Corollary:* Φ_* is an exact functor on the category of quasi-coherent $\mathcal{O}_X$-modules [Compare Liu, 5.1.8].

(1.6.5) *Proposition:* Let N' be an A'-module and set $N = N' \otimes_{A'} A$. Then there is a natural isomorphism $\Phi^*(\widetilde{N'}) \cong \widetilde{N}$.

(1.6.6) *Corollary:* The sections of $\Phi^*(\widetilde{N'})$ induced by global sections of $\widetilde{N'}$ generate $\Gamma(X, \Phi^* \widetilde{N'})$.

(1.6.7) In the setting of (1.6.5), the canonical homomorphism $\widetilde{N'} \to \Phi_*\Phi^*(\widetilde{N'})$ is $\widetilde{j}$, where $j \colon N' \to N' \otimes_{A'} A_{[\phi]}$ is given by $z' \mapsto z' \otimes 1$. Similarly, the canonical homomorphism $\Phi^*\Phi_*(\widetilde{M}) \to \widetilde{M}$ is $\widetilde{p}$, where $p \colon M_{[\phi]} \otimes_{A'} A \to M$ is given by $z \otimes a \mapsto az$.

It follows that if $v \colon N' \to M_{[\phi]}$ is an A'-module homomorphism, then $\widetilde{v}^\sharp = (v \otimes 1)^\sim$.

(1.6.8) Let N'_1, N'_2 be A'-modules, with N'_1 finitely presented. Then the canonical homomorphism

$$\Phi^*(\mathcal{H}om_{\mathcal{O}_{X'}}(\widetilde{N'_1}, \widetilde{N'_2})) \to \mathcal{H}om_{\mathcal{O}_X}(\Phi^*(\widetilde{N'_1}), \Phi^*(\widetilde{N'_2}))$$

is $\widetilde{\gamma}$, where γ is the canonical A-module homomorphism $\operatorname{Hom}_{A'}(N'_1, N'_2) \otimes_{A'} A \to \operatorname{Hom}_A(N'_1 \otimes_{A'} A, N'_2 \otimes_{A'} A)$.

(1.6.9) Let $\mathcal{I}'$ be an ideal of A' and M an A-module. Then $\widetilde{\mathcal{I}'}\widetilde{M}$ (which by definition means the image of $\Phi^*(\widetilde{\mathcal{I}'}) \otimes_{\mathcal{O}_X} \widetilde{M} \to \widetilde{M}$) is identified canonically with $(\mathcal{I}'M)^\sim$. In particular, taking $M = A$ and using the right exactness of Φ^*, the sheaf of $\mathcal{O}_X$-algebras $\Phi^*((A'/\mathcal{I}')^\sim)$ is identified with $(A/\mathcal{I}'A)^\sim$.

(1.6.10) Given a third ring A'' and $\phi' : A'' \to A'$, let $\phi'' = \phi \circ \phi'$. Then $\Phi'' = \Phi' \circ \Phi$, that is, $A \to (\operatorname{Spec}(A), \widetilde{A})$ is a contravariant functor from commutative rings to ringed spaces.