

SYNOPSIS OF MATERIAL FROM EGA CH. I

§1: AFFINE SCHEMES, 1.1–1.3

1.1 Prime spectrum of a ring. [Liu, 2.1 and 2.3]

(1.1.1) Notation: $X = \text{Spec}(A) = \{\text{prime ideals of } A\}$. We write \mathfrak{i}_x for $x \in X$ to emphasize its role as an ideal of A . We have $\text{Spec}(A) = \emptyset$ iff $A = 0$.

$A_x = A_{\mathfrak{i}_x} =$ local ring.

$\mathfrak{m}_x = \mathfrak{i}_x A_x =$ maximal ideal of A_x .

$k(x) = A_x/\mathfrak{m}_x =$ residue field of A_x , equal to the field of fractions of A/\mathfrak{i}_x .

$f(x) =$ image in $k(x)$ of $f \in A$, so $f(x) = 0$ iff $f \in \mathfrak{i}_x$.

$M_x = M \otimes_A A_x =$ localization of an A -module M .

$\sqrt{(E)} =$ radical of the ideal generated by $E \subseteq A$.

$V(E) = \{x \in X : E \subseteq \mathfrak{i}_x\} = \{x \in X : f(x) = 0 \text{ for all } f \in E\}$.

(1.1.1.1) *Lemma* [Liu, 2.1.6]: $\sqrt{(E)} = \bigcap_{x \in V(E)} \mathfrak{i}_x$.

$V(f) = V(\{f\})$ for $f \in A$.

$D(f) = X \setminus V(f) = \{x \in X : f(x) \neq 0\}$.

(1.1.2) *Proposition:* One has

(i) $V(0) = X$, $V(1) = \emptyset$.

(ii) $E \subseteq E' \Rightarrow V(E) \supseteq V(E')$.

(iii) For any collection (E_λ) , $V(\bigcup_\lambda E_\lambda) = \bigcap_\lambda V(E_\lambda)$.

(iv) $V(E E') = V(E) \cup V(E')$.

(v) $V(E) = V(\sqrt{(E)})$.

The sets $V(E)$ are the closed subsets of a topology, the *Zariski topology* on X . We understand $X = \text{Spec}(A)$ to have this topology from now on.

(1.1.3) Given $Y \subseteq X$, let $\mathfrak{i}(Y) = \{f \in A : f(x) = 0 \text{ for all } x \in Y\} = \bigcap_{x \in Y} \mathfrak{i}_x$. Then $Y \subseteq Y' \Rightarrow \mathfrak{i}(Y) \supseteq \mathfrak{i}(Y')$ and

$$\mathfrak{i}\left(\bigcup_\lambda Y_\lambda\right) = \bigcap_\lambda \mathfrak{i}(Y_\lambda).$$

Also, $\mathfrak{i}(\{x\}) = \mathfrak{i}_x$.

(1.1.4) *Proposition:*

(i) For any $E \subseteq A$, $\mathfrak{i}(V(E)) = \sqrt{(E)}$.

(ii) For any $Y \subseteq X$, $V(\mathfrak{i}(Y)) = \overline{Y}$ is the closure of Y .

(1.1.5) *Corollary:* Closed subsets $Y \subseteq X$ and radical ideals $\mathfrak{a} \subseteq A$ correspond bijectively via $Y \mapsto \mathfrak{i}(Y)$, $\mathfrak{a} \mapsto V(\mathfrak{a})$; the union $Y_1 \cup Y_2$ corresponding to $\mathfrak{i}(Y_1) \cap \mathfrak{i}(Y_2)$ and an arbitrary intersection $\bigcap_\lambda Y_\lambda$ corresponding to $\sqrt{\sum_\lambda \mathfrak{i}(Y_\lambda)}$.

(1.1.6) *Corollary:* If A is a Noetherian ring, then $\text{Spec}(A)$ is a Noetherian space (the converse is false).

(1.1.7) *Corollary:* The closure of $\{x\}$ is the set of $y \in X$ such that $\mathfrak{i}_x \subseteq \mathfrak{i}_y$. Thus $\{x\}$ is closed iff \mathfrak{i}_x is maximal.

(1.1.8) *Corollary:* $\text{Spec}(A)$ is a T_0 space.

(1.1.9-10) *Proposition:* (i) $D(f) \cap D(g) = D(fg)$, and as f runs through A , the sets $D(f)$ form a base of the topology on $\text{Spec}(A)$.

(ii) $D(f)$ is quasi-compact, and in particular, so is $X = D(1)$.

(1.1.11) *Proposition:* $\text{Spec}(A/\mathfrak{a})$ is canonically identified with the closed subset $V(\mathfrak{a}) \subseteq \text{Spec}(A)$.

(1.1.12) *Corollary:* $\text{Spec}(A)$ and $\text{Spec}(A/\sqrt{(0)})$ are canonically homeomorphic.

(1.1.13) *Proposition:* $\text{Spec}(A)$ is an irreducible space iff $A/\sqrt{(0)}$ is a domain, *i.e.*, $\sqrt{(0)}$ is prime.

(1.1.14) *Corollary:* (i) In the correspondence between closed subsets of X and radical ideals of A , the irreducible closed subsets correspond to the prime ideals.

(ii) $x \mapsto \overline{\{x\}}$ is a bijection from points of X to irreducible closed subsets of X , *i.e.*, each irreducible closed subset has a unique generic point.

(1.1.15) *Proposition:* If \mathfrak{a} is an ideal contained in the Jacobson radical $\mathfrak{R}(A)$, the unique open neighborhood of $V(\mathfrak{a})$ is X itself.

1.2 Functorial properties of Spec .

(1.2.1) A ring homomorphism $\phi: A' \rightarrow A$ induces a map

$${}^a\phi: X = \text{Spec}(A) \rightarrow X' = \text{Spec}(A')$$

by ${}^a\phi(x) = \phi^{-1}(\mathfrak{i}_x)$. We denote by ϕ^x the induced injection $A'/\phi^{-1}(\mathfrak{i}_x) \rightarrow A/\mathfrak{i}_x$, as well as its extension to their fraction fields

$$\phi^x: k({}^a\phi(x)) \rightarrow k(x).$$

For any $f' \in A'$, we have $\phi^x(f'({}^a\phi(x))) = (\phi(f'))(x)$.

(1.2.2) *Proposition:* (i) For any $E' \subseteq A'$, we have ${}^a\phi^{-1}(V(E')) = V(\phi(E'))$; in particular ${}^a\phi^{-1}(D(f')) = D(\phi(f'))$.

(ii) For any ideal $\mathfrak{a} \subseteq A$, we have $\overline{{}^a\phi(V(\mathfrak{a}))} = V(\phi^{-1}(\mathfrak{a}))$.

(1.2.3) *Corollary:* ${}^a\phi$ is continuous.

In fact, Spec is a contravariant functor from commutative rings to topological spaces.

(1.2.4) *Corollary* [Liu, 2.1.7 (b)]: Suppose that ϕ is surjective, or more generally, that every $f \in A$ has the form $f = h\phi(f')$ for some $f' \in A'$ and invertible $h \in A$. Then ${}^a\phi$ is a homeomorphism from X onto ${}^a\phi(X)$.

(1.2.5) In particular, when ϕ is the canonical homomorphism $A \rightarrow A/\mathfrak{a}$, then ${}^a\phi$ is the inclusion of $\text{Spec}(A/\mathfrak{a})$, identified with $V(\mathfrak{a})$, into $\text{Spec}(A)$.

(1.2.6) *Corollary* [Liu, 2.1.7 (c)]: For any multiplicative set $S \subseteq A$, $\text{Spec}(S^{-1}A)$ is canonically homeomorphic to the subspace $\{x \in X : \mathfrak{i}_x \cap S = \emptyset\}$ of $X = \text{Spec}(A)$.

(1.2.7) *Corollary:* ${}^a\phi(X)$ is dense in X' iff $\ker(\phi)$ is nilpotent.

1.3 Sheaf associated to a module. [Liu, Section 5.1.2]

(1.3.1) Let M be an A -module, $f \in A$, $S_f = \{1, f, f^2, \dots\}$, $A_f = S_f^{-1}A$, $M_f = S_f^{-1}M$. Let $S'_f = \{g \in A : g \text{ divides } f^n \text{ for some } n\}$ be the saturation of S_f , so $S'^{-1}_f A = S_f^{-1}A$, $S'^{-1}_f M = S_f^{-1}M$ canonically (0, 1.4.3).

(1.3.2) *Lemma:* The following are equivalent: (a) $g \in S'_f$, (b) $S'_g \subseteq S'_f$, (c) $f \in \sqrt{(g)}$, (d) $\sqrt{(f)} \subseteq \sqrt{(g)}$, (e) $V(g) \subseteq V(f)$, (f) $D(f) \subseteq D(g)$.

(1.3.3) If $D(f) = D(g)$ it follows that $M_f = M_g$, and for $D(f) \supseteq D(g)$ there is a canonical homomorphism, functorial in M ,

$$\rho_{g,f}: M_f \rightarrow M_g,$$

satisfying $\rho_{h,g} \circ \rho_{g,f}$ for $D(f) \supseteq D(g) \supseteq D(h)$. Given $x \in \text{Spec}(A)$, the localization M_x is the direct limit $\varinjlim M_f$ of the system formed by the modules M_f and homomorphisms $\rho_{g,f}$, as f varies over $A \setminus \mathfrak{i}_x$. Write

$$\rho_x^f: M_f \rightarrow M_x$$

for the canonical homomorphism, for $f \in A \setminus \mathfrak{i}_x$ (i.e., $x \in D(f)$).

(1.3.4) *Definition:* The *structure sheaf* of $X = \text{Spec}(A)$ (resp. *sheaf associated to an A -module M*), denoted \widetilde{A} or \mathcal{O}_X (resp. \widetilde{M}) is the sheaf of rings (resp. sheaf of \mathcal{O}_X -modules) associated to the presheaf $D(f) \mapsto A_f$ (resp. $D(f) \mapsto M_f$) on the base \mathcal{B} of open sets of the form $D(f)$.

[In (1.3.7) it will be shown that this presheaf on \mathcal{B} is already a sheaf, so “sheaf associated to” is superfluous here.]

The stalks \widetilde{A}_x (resp. \widetilde{M}_x) are just A_x (resp. M_x), and we have canonical homomorphisms $\theta_f: A_f \rightarrow \Gamma(D(f), \widetilde{A})$ and similarly for M , such that $(\theta_f(m))_x = \rho_x^f(m)$ for $m \in M_f$.

(1.3.5) *Proposition* [Liu, 5.1.5 (b)]: $M \mapsto \widetilde{M}$ is an exact, contravariant functor from A -modules to sheaves of \widetilde{A} -modules.

(1.3.6) *Proposition* [Liu, 2.3.7]: For every $f \in A$, the open set $D(f)$ is canonically homeomorphic to $\text{Spec}(A_f)$, and the sheaf \widetilde{M}_f associated to the A_f -module M_f coincides under this identification with $\widetilde{M}|_{D(f)}$.

(1.3.7) *Theorem* [Liu, 2.3.1 (a)]: The canonical homomorphism $\theta_f: M_f \rightarrow \Gamma(D(f), \widetilde{M})$ is an isomorphism. In particular, $M \cong \Gamma(X, \widetilde{M})$.

(1.3.8) *Corollary:* Given two A -modules M, N , the functorial map $\text{Hom}_A(M, N) \rightarrow \text{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is bijective, i.e., the functor $M \mapsto \widetilde{M}$ is fully faithful. In particular $\widetilde{M} = 0$ iff $M = 0$.

(1.3.9) *Corollary:* (i) Given $u: M \rightarrow N$, the sheaves associated to $\ker u$, $\text{im } u$, $\text{coker } u$ are $\ker \widetilde{u}$, $\text{im } \widetilde{u}$, $\text{coker } \widetilde{u}$. Thus \widetilde{u} is injective (resp. surjective, bijective) iff u is.

(ii) The functor $M \mapsto \widetilde{M}$ commutes with all inductive limits, in particular with all direct sums [Liu, 5.1.5 (a)].

Note that the sheaves associated to A -modules form an Abelian category, by (1.3.8). If M is finitely generated, then there is a surjection $\widetilde{A}^n \rightarrow \widetilde{M}$, so \widetilde{M} is generated by a finite family of global sections.

(1.3.10) The functor $M \rightarrow \widetilde{M}$ preserves (finite) intersections and sums of submodules

(1.3.11) *Corollary:* On the category of sheaves associated to A -modules, the global section functor Γ is exact.

(1.3.12) *Corollary:* (i) $M \mapsto \widetilde{M}$ commutes with (finite) tensor products [Liu, 5.1.5 (d)].

(ii) For finitely presented M , the sheaf associated to $\mathrm{Hom}_A(M, N)$ is canonically identified with $\mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$.

(1.3.13) If B is an A -algebra, then \widetilde{B} is a sheaf of \widetilde{A} -algebras, and if M is a B -module, then \widetilde{M} is a sheaf of \widetilde{B} -modules, of finite type iff M is a finitely generated B -module. The results in (1.3.8-12) apply in this setting as well. If B (resp. M) is graded, then so is \widetilde{B} (resp. \widetilde{M}) [cf. (0, 4.1.4)].

(1.3.14) If B is an A -algebra, $M \subseteq B$ an A -submodule, and $C \subseteq B$ the sub-algebra generated by M , then \widetilde{C} is the sub- \widetilde{A} -algebra of \widetilde{B} generated by \widetilde{M} [cf. (0, 4.1.3)].