Synopsis of material from EGA Ch. 0
§5: Coherent and quasi-coherent sheaves

5.1 Quasi-coherent sheaves.

(5.1.1) Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. To give a global section $s \in \mathcal{F}(X)$ it is equivalent to give the $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to \mathcal{F}$ which sends $1 \in \mathcal{O}_X(X)$ to s. To give a family of global sections $(s_i)_{i \in I}$ is then equivalent to giving a homomorphism $\mathcal{O}_X^{(I)} \to \mathcal{F}$, where $\mathcal{O}_X^{(I)}$ is the direct sum of copies of $\mathcal{O}_X$ corresponding to the indices $i \in I$. If this homomorphism is surjective, $\mathcal{F}$ is said to be *generated* by the global sections (s_i) [Liu, 5.1.2-3].

One can give an example of a sheaf $\mathcal{F}$ and a point x such that $\mathcal{F}|U$ is not generated by its sections on U for *any* neighborhood U of x.

(5.1.2) Let $f\colon X \to Y$ be a morphism of ringed spaces. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules which is generated by global sections, then the canonical homomorphism $f^*(f_*\mathcal{F}) \to \mathcal{F}$ is surjective (but not conversely). If $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules which is generated by global sections, then $f^*\mathcal{G}$ is generated by global sections.

(5.1.3) A sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules is *quasi-coherent* if every $x \in X$ has a neighborhood U such that $\mathcal{F}|U$ is isomorphic to the cokernel of a homomorphism $\mathcal{O}_X^{(J)} \to \mathcal{O}_X^{(I)}$ [Liu, 5.1.4]. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is *quasi-coherent* if it is so as an $\mathcal{O}_X$-module. Note that $\mathcal{O}_X$ and any direct sum $\mathcal{O}_X^{(I)}$ are quasi-coherent.

(5.1.4) Given $f\colon X \to Y$ and $\mathcal{G}$ a quasi-coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is quasi-coherent.

5.2 Finitely generated sheaves.

(5.2.1) An $\mathcal{O}_X$-module $\mathcal{F}$ is *finitely generated* [Liu, 5.1.10] if every $x \in X$ has a neighborhood U such that $\mathcal{F}|U$ is generated by finitely many sections on U. Equivalently, $\mathcal{F}|U$ is a quotient of $(\mathcal{O}_X|U)^p$ for an integer $p > 0$.

A quotient of a finitely generated sheaf is finitely generated. A finitely generated sheaf is not necessarily quasi-coherent. If $\mathcal{F}$ is finitely generated, then its stalks $\mathcal{F}_x$ are finitely generated $\mathcal{O}_x$-modules, but the converse need not hold.

(5.2.2) Let $\mathcal{F}$ be finitely generated. The support of $\mathcal{F}$ is closed, since it is locally the union of the supports of generating sections. If $u\colon \mathcal{F} \to \mathcal{G}$ is a sheaf homomorphism, the set of points x where $u_x = 0$ is open.

(5.2.3) Assume X is quasi-compact, $u\colon \mathcal{F} \to \mathcal{G}$ surjective, and $\mathcal{G}$ finitely generated. If $\mathcal{F} = \varinjlim_\lambda(\mathcal{F}_\lambda)$ is a direct limit, then some $\mathcal{F}_\lambda \to \mathcal{G}$ is surjective.

Again, if X is quasi-compact, and $\mathcal{F}$ is finitely generated and generated by global sections, then $\mathcal{F}$ is generated by finitely many global sections.

(5.2.4) Given $f\colon X \to Y$, if $\mathcal{G}$ is a finitely generated $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is finitely generated.

(5.2.5) An $\mathcal{O}_X$-module $\mathcal{F}$ is *finitely presented* if every $x \in X$ has a neighborhood U such that $\mathcal{F}|U$ is isomorphic to the cokernel of a homomorphism $\mathcal{O}_X^q \to \mathcal{O}_X^p$. In particular, $\mathcal{F}$ is then finitely generated and quasi-coherent. Given $f\colon X \to Y$, if $\mathcal{G}$ is a finitely presented $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is finitely presented.

(5.2.6) If $\mathcal{F}$ is finitely presented, then the canonical homomorphism

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$$

is an isomorphism.

(5.2.7) If $\mathcal{F}$ and $\mathcal{G}$ are both finitely presented, and $\mathcal{F}_x$ is isomorphic to $\mathcal{G}_x$, there is an open neighborhood U of x such that $\mathcal{F}|U$ and $\mathcal{G}|U$ are isomorphic.

5.3 Coherent sheaves.

[This concept is treated in Liu, Section 5.1.3, under the simplifying hypothesis that $(X,\mathcal{O}_X)$ is a locally Noetherian scheme. In that case "finitely generated," "finitely presented," and "coherent" are equivalent.]

(5.3.1) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *coherent* [Liu, 5.1.10] if

(a) $\mathcal{F}$ is finitely generated, and

(b) For every open set U and every homomorphism $u\colon \mathcal{O}_X^n|U \to \mathcal{F}|U$, $\ker(u)$ is finitely generated.

(5.3.2) A coherent $\mathcal{O}_X$-module is finitely presented, but not conversely; indeed $\mathcal{O}_X$ itself may not be coherent. A finitely generated sub-$\mathcal{O}_X$-module of a coherent sheaf is coherent. A finite direct sum of coherent sheaves is coherent.

(5.3.3) For any exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, if two of the terms are coherent, then so is the third.

(5.3.4) The kernel, image and cokernel of a homomorphism between coherent sheaves are coherent. In particular, the sum and intersection of two coherent subsheaves of a coherent sheaf are coherent.

(5.3.5) If $\mathcal{F}$ and $\mathcal{G}$ are coherent, then so are $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ and $\mathcal{H}om(\mathcal{F},\mathcal{G})$.

(5.3.6) If $\mathcal{I} \subseteq \mathcal{O}_X$ is a coherent sheaf of ideals, and $\mathcal{F}$ is a coherent sheaf, then $\mathcal{I}\mathcal{F}$ is coherent.

(5.3.7) An $\mathcal{O}_X$-algebra is *coherent* if it is coherent as an $\mathcal{O}_X$-module. In particular, $\mathcal{O}_X$ is a *coherent sheaf of rings* if it is coherent as a module over itself, *i.e.*, every homomorphism $u\colon \mathcal{O}_X^n|U \to \mathcal{O}_X|U$ has finitely generated kernel.

If $\mathcal{O}_X$ itself is coherent, then a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is finitely presented.

The *annihilator* of $\mathcal{F}$ is the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ given by the kernel of the canonical homomorphism $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$. If $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, then $\mathcal{I}$ is coherent and for all $x \in X$, $\mathcal{I}_x$ is the annihilator of $\mathcal{F}_x$ in $\mathcal{O}_x$.

(5.3.8) Assume $\mathcal{O}_X$ and $\mathcal{F}$ are coherent. Given any $x \in X$ and any finitely generated submodule $M \subseteq \mathcal{F}_x$, there is a neighborhood U of x and a coherent subsheaf $\mathcal{G} \subseteq \mathcal{F}|U$ such that $\mathcal{G}_x = M$.

Note that the above implies certain necessary conditions on the stalks $\mathcal{O}_x$ for $\mathcal{O}_X$ to be coherent. For example, any intersection of finitely generated ideals of $\mathcal{O}_x$ must be finitely generated.

(5.3.9) If $\mathcal{O}_X$ is coherent and M is a finitely presented $\mathcal{O}_x$-module, there is a neigborhood U of x and a coherent sheaf of $(\mathcal{O}_X|U)$-modules $\mathcal{F}$, such that $\mathcal{F}_x = M$.

(5.3.10) Assume $\mathcal{O}_X$ is coherent and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent sheaf of ideals. Then a sheaf of $(\mathcal{O}_X/\mathcal{I})$-modules is coherent iff it is so as a sheaf of $\mathcal{O}_X$-modules. In particular, $\mathcal{O}_X/\mathcal{I}$ itself is coherent.

(5.3.11) Given $f\colon X \to Y$, if $\mathcal{O}_X$ is coherent, and $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is coherent.

(5.3.12) Let $j\colon Y \hookrightarrow X$ be the inclusion of a closed subset Y. Let $\mathcal{O}_Y$ be a coherent sheaf of rings on Y. Then a sheaf $\mathcal{G}$ of $\mathcal{O}_Y$ modules is finitely generated, quasi-coherent, or coherent iff $j_*\mathcal{G}$ has the same property as a sheaf of $j_*\mathcal{O}_Y$-modules.

5.4 Locally free sheaves.

(5.4.1) [Liu, Exercise 5.1.12] A sheaf of $\mathcal{O}_X$-modules is *locally free* if every $x \in X$ has a neighborhood U such that $\mathcal{F}|U \cong \mathcal{O}_X^{(I)}|U$ for some index set I (which may depend on U). If I is finite for all U, $\mathcal{F}$ has *finite rank*; if $|I| = n$ for all U, $\mathcal{F}$ is *of rank n*. An *invertible sheaf* is a locally free sheaf of rank 1 [Liu, 5.1.21].

If $\mathcal{F}$ is locally free of finite rank, then $\mathcal{F}_x$ is a finitely generated free $\mathcal{O}_x$-module of rank $n(x)$, and $\mathcal{F}|U$ is of rank $n(x)$ on some neighborhood U of x. In particular, if X is connected, the rank $n(x)$ is constant.

A locally free sheaf is quasi-coherent. If $\mathcal{O}_X$ is coherent, then a locally free sheaf of finite rank is coherent.

If $\mathcal{F}$ is locally free, then $- \otimes_{\mathcal{O}_X} \mathcal{F}$ is an exact functor (*i.e.*, $\mathcal{F}$ is *flat*).

The term "locally free sheaf" will be understood as implying finite rank unless otherwise specified.

(5.4.2) Let $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. We have a canonical homomorphism

$$\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{F}).$$

If $\mathcal{L}$ is locally free of finite rank, this is an isomorphism.

(5.4.3) If $\mathcal{L}$ is invertible, then so is its dual $\mathcal{L}^\vee$, and

$$\mathcal{L}^\vee \otimes_{\mathcal{O}} \mathcal{L} \cong \mathcal{O}_X$$

For this reason we write $\mathcal{L}^{-1} = \mathcal{L}^\vee$ in this case. If $\mathcal{L}$ is invertible, then $\mathcal{L}_x \cong \mathcal{O}_x$ as an $\mathcal{O}_x$ module for all $x \in X$. If we assume that $\mathcal{O}_x$ and $\mathcal{L}$ are coherent, then this condition is sufficient for $\mathcal{L}$ to be invertible.

(5.4.4) If $\mathcal{L}$, $\mathcal{L}'$ are invertible, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$. Define $\mathcal{L}^{\otimes n}$ to be the n-th tensor power of $\mathcal{L}$ for $n > 0$, and set $\mathcal{L}^{\otimes 0} = \mathcal{O}_X$, $\mathcal{L}^{\otimes(-n)} = (\mathcal{L}^{-1})^{\otimes n}$. Then

$$\mathcal{L}^{\otimes m} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} = \mathcal{L}^{\otimes(m+n)}$$

for all integers m, n.

(5.4.5) Given $f\colon Y \to X$, if $\mathcal{L}$ is a locally free (resp. invertible) sheaf of $\mathcal{O}_X$-modules, then so is $f^*\mathcal{L}$. We have $f^*(\mathcal{L}^\vee) = (f^*\mathcal{L})^\vee$ and $f^*(\mathcal{L}^{\otimes n}) = (f^*\mathcal{L})^{\otimes n}$.

(5.4.6) Let $\mathcal{L}$ be invertible. Then $\Gamma_*(\mathcal{L}) \underset{def}{=} \sum_{n\in\mathbb{Z}} \Gamma(X, \mathcal{L}^{\otimes n})$ is a graded ring, the product of sections $s_m \in \Gamma(X, \mathcal{L}^{\otimes m})$, $s_n \in \Gamma(X, \mathcal{L}^{\otimes n})$ being given by $s_m \otimes s_n$. Clearly Γ_* is a covariant functor of $\mathcal{L}$.

For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we then define

$$\Gamma_*(\mathcal{L}, \mathcal{F}) = \sum_n \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}),$$

and this is a graded $\Gamma_*(\mathcal{L})$-module, functorial in $\mathcal{F}$ for fixed $\mathcal{L}$.

Given $f : Y \to X$, we have a canonical homomorphism of graded rings $\Gamma_*(\mathcal{L}) \to \Gamma_*(f^*\mathcal{L})$, which is a natural transformation of functors in $\mathcal{L}$. Similarly we have a homomorphism of abelian groups $\Gamma_*(\mathcal{L}, \mathcal{F}) \to \Gamma_*(f^*\mathcal{L}, f^*\mathcal{F})$, which together with $\Gamma_*(\mathcal{L}) \to \Gamma_*(f^*\mathcal{L})$ gives a di-homomorphism of graded rings and graded modules.

(5.4.7-8) It can be shown that the isomorphism classes of invertible $\mathcal{O}_X$-modules form a set, which is then a group under $\otimes$. This group [the *Picard group* $\mathrm{Pic}(X)$, cf. Liu, Exercise 5.2.7] is isomorphic to the sheaf cohomology group $H^1(X, \mathcal{O}_X^*)$ where $\mathcal{O}_X^* \subseteq \mathcal{O}_X$ is the subsheaf such that $\mathcal{O}_X^*(U)$ is the multiplicative group of invertible elements of $\mathcal{O}_X(U)$ (so $\mathcal{O}_X^*$ is a sheaf of abelian groups). If $f\colon Y \to X$ is a homomorphism of ringed spaces, then f^* gives a homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$, which agrees with the homomorphism $H^1(X, \mathcal{O}_X^*) \to H^1(Y, \mathcal{O}_Y^*)$ of sheaf cohomology groups induced by f.

[EGA goes on to give the proof, which we will discuss this later, after we introduce sheaf cohomology.]

(5.4.9) Any exact sequence $0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{F} \to 0$ splits locally if $\mathcal{F}$ is locally free.

(5.4.10) Given $f\colon X \to Y$, any $\mathcal{O}_X$-module $\mathcal{F}$, and a locally free $\mathcal{O}_Y$-module $\mathcal{L}$ of finite rank, there is a canonical isomorphism

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L} \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})).$$

5.5 Sheaves on a locally ringed space.

(5.5.1) $(X, \mathcal{O}_X)$ is a *locally ringed space* if each stalk $\mathcal{O}_x$ is a local ring [Liu only considers locally ringed spaces; cf. 2.2.19]. Notation: $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$; $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field; if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, $f \in \mathcal{O}_X(U)$ and $x \in U$, then $f(x)$ denotes the image of the germ $f_x \in \mathcal{F}_x$ in the $k(x)$-vector space $\mathcal{F}_x/m_x\mathcal{F}_x = \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)$. In particular, $f(x) = 0$ means $f_x \in m_x\mathcal{F}_x$, but does not necessarily mean $f_x = 0$.

(5.5.2) Let X be a locally ringed space, $\mathcal{L}$ an invertible sheaf, and $f \in \mathcal{L}(X)$ a global section. The following are equivalent:

(a) f_x generates $\mathcal{L}_x$;

(b) $f_x \notin m_x\mathcal{L}_x$, that is, $f(x) \neq 0$;

(c) there is a section $g \in \mathcal{L}^{-1}(U)$ for some neighborhood U of x such that $f \otimes g = 1 \in \mathcal{O}_X(U)$.

By (c), the set X_f of points x where these conditions hold is open; it is called the non-vanishing locus of f.

(5.5.3) Keep the hypotheses of (5.5.2) and let $\mathcal{L}'$ be another invertible sheaf, $g \in \mathcal{L}'(X)$. Then $X_f \cap X_g = X_{f \otimes g}$.

(5.5.4) If $\mathcal{F}$ is locally free of rank n, then $\bigwedge^p \mathcal{F}$ is locally free of rank $\binom{n}{p}$, and we have a canonical isomorphism of $k(x)$-vector spaces $(\bigwedge^p \mathcal{F})_x/\mathfrak{m}_x(\bigwedge^p \mathcal{F})_x \cong \bigwedge^p(\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x)$. Let $s_1, \ldots, s_p \in \mathcal{F}(U)$, and set $s = s_1 \wedge \cdots \wedge s_p \in (\bigwedge^p \mathcal{F})(U)$. One says that $s_1, \ldots, s_p$ are *linearly dependent* if $s = 0$, and that $s_1(x), \dots, s_p(x)$ are linearly dependent if $s(x) = 0$. The set of points $x \in U$ where this holds is open.

Given n sections $s_1, \dots, s_n \in \mathcal{F}(U)$ such that $s_1(x), \dots, s_n(x)$ are linearly independent for all $x \in U$, they define an *isomorphism* $u \colon \mathcal{O}_X^n|U \to \mathcal{F}|U$.

(5.5.5) Let $u \colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of locally free sheaves of finite rank. The following are equivalent:

(a) u_x induces an *injective* map of $k(x)$-vector spaces $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \hookrightarrow \mathcal{G}_x/\mathfrak{m}_x\mathcal{G}_x$;

(b) there is a neighborhood U of X such that $u|U$ is injective, and the quotient $(\mathcal{G}/(\mathrm{im}(u))|U$ is locally free (and hence, after possibly replaceing U be a smaller neighborhood, the exact sequence $0 \to \mathcal{F}|U \underset{u}{\to} \mathcal{G}|U \to (\mathcal{G}/\operatorname{im}(u))|U \to 0$ splits).