SYNOPSIS OF MATERIAL FROM EGA CH. 0
§3: SUPPLEMENT ON SHEAVES
§4: RINGED SPACES

For an introduction to sheaves and ringed spaces, see Liu chapter 2.2.

3.1 Sheaves with values in a category.

(3.1.1-4) Suppose given a collection of objects A_α and morphisms $\rho_i\colon A_{\alpha_i} \to A_{\beta_i}$ in a category K. A *projective limit* of the system (A_α, ρ_i) is an object X, equipped with morphisms $\rho_\alpha\colon X \to A_\alpha$ commuting with all the morphisms ρ_i, such that for every such object X' with morphisms $\rho'_\alpha\colon X' \to A_\alpha$, there is a unique morphism $\phi\colon X' \to X$ such that $\rho'_\alpha = \rho_\alpha \circ \phi$ for all α. As with any object characterized by a universal property, a projective limit X is unique up to canonical isomorphism if it exists.

The open subsets of a topological space X form a small category $\mathcal{U}$, with morphisms the inclusions $U \subseteq V$. A *presheaf with values in a category* K is a contravariant functor $\mathcal{F}\colon \mathcal{U} \to K$. The morphisms $\rho^U_V\colon \mathcal{F}(U) \to \mathcal{F}(V)$ which $\mathcal{F}$ associates to inclusions $V \subseteq U$ are called *restriction morphisms*. $\mathcal{F}$ is a *sheaf* if it satisfies the axiom:

(F) For every open cover $U = \bigcup_\alpha U_\alpha$, $\mathcal{F}(U)$ is the projective limit of the system given by the objects $\mathcal{F}(U_\alpha)$ and $\mathcal{F}(U_\alpha \cap U_\beta)$ and the restriction morphisms $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}$.

These definitions reduce to the usual ones [Liu, 2.2.2; see also 2.2.7] when K is the category of sets, or more generally when K is a category of sets equipped with algebraic structure, such as rings, abelian groups, etc.. In that case K admits all projective limits, which coincide with projective limits of the underlying sets.

Suppose, however, that K is for example the category of topological rings (with continuous ring homomorphisms). Then a sheaf $\mathcal{F}$ with values in K is a sheaf of rings in the usual sense, but with the additional requirement that for every open cover $U = \bigcup_\alpha U_\alpha$, the ring $\mathcal{F}(U)$ carries the coarsest topology such that all the restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_\alpha)$ are continuous.

(3.1.5) [Liu, 2.2.5] If $\mathcal{F}$ is a presheaf (resp. sheaf) with values in K and $U \subseteq X$ is an open set, the $\mathcal{F}(V)$ for open sets $V \subseteq U$ form a presheaf (resp. sheaf) on U, denoted $\mathcal{F}|U$. The restriction $\mathcal{F} \mapsto \mathcal{F}|U$ is a functor.

(3.1.6) If the category K admits *inductive limits* (the concept dual to that of projective limit defined above), one defines the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at $x \in X$ to be the inductive limit of the $\mathcal{F}(U)$ for all neighborhoods $x \in U$.

If K is a category of sets with algebraic structure (rings, abelian groups, $\ldots$), then $\mathcal{F}_x$ is the direct limit, and we use the notation s_x for the germ in $\mathcal{F}_x$ of a section $s \in \mathcal{F}(U)$, where $x \in U$, as in [Liu, 2.2.8]. The *support* of a sheaf of abelian groups, modules or rings is defined as in [Liu, Exercise 2.2.5]

3.2 Presheaves on a base of open sets. [Liu, 2.2.6]

(3.2.1) Assume K admits all projective limits. Let $\mathfrak{B}$ be a base of open subsets on X. Regarding $\mathfrak{B}$ as a category, with morphisms given by inclusions $U \subseteq V$, a *presheaf on* $\mathfrak{B}$ *with values in* K is a contravariant functor $\mathcal{F}\colon \mathfrak{B} \to K$. For any open set U, the objects $\mathcal{F}(V)$

for $V \in \mathfrak{B}$ and $V \subseteq U$ form a projective system. Let $\mathcal{F}'(U) = \varprojlim \mathcal{F}(V)$ be its projective limit. Then $\mathcal{F}'$ is a presheaf on X with values in K, and for $U \in \mathfrak{B}$ one can identify $\mathcal{F}'(U)$ with $\mathcal{F}(U)$. (If X is a Noetherian space, it is possible to define $\mathcal{F}'(U)$ assuming only that K admits finite projective limits.)

(3.2.2) The necessary and sufficient condition for $\mathcal{F}'$ defined above to be a *sheaf* is:

(F_0) For every $U \in \mathfrak{B}$ and open cover $U = \bigcup_\alpha U_\alpha$ with all $U_\alpha \in \mathfrak{B}$, $\mathcal{F}(U)$ is the projective limit of the system given by the objects $\mathcal{F}(U_\alpha)$ and $\mathcal{F}(U_\alpha \cap U_\beta)$ and the restriction morphisms $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}$.

[Given an object $T \in K$, we can define a presheaf of sets $\mathcal{G}(U) = \operatorname{Hom}_K(T, \mathcal{F}(U))$ on $\mathfrak{B}$ by composing the functor $\operatorname{Hom}(T, -)$ with $\mathcal{F}$. The condition that $\mathcal{F}(U)$ is the projective limit of the system given by the objects $\mathcal{F}(U_\alpha)$ and $\mathcal{F}(U_\alpha \cap U_\beta)$ is then equivalent to the condition that for every $T \in K$, the sets $\mathcal{G}(U)$ satisfy the usual set-theoretic formulation of the sheaf axiom [Liu, 2.2.2]. In other words, if we denote the restriction maps by $\rho_\alpha \colon \mathcal{G}(U) \to \mathcal{G}(U_\alpha)$ (resp. $\rho_{\alpha\beta} \colon \mathcal{G}(U_\alpha) \to \mathcal{G}(U_\alpha \cap U_\beta)$), then for every system of elements $(s_\alpha) \in \prod_\alpha \mathcal{G}(U_\alpha)$ satisfying $\rho_{\alpha\beta}(s_\alpha) = \rho_{\beta\alpha}(s_\beta)$ for all α, β, there is a unique $s \in \mathcal{G}(U)$ such that $s_\alpha = \rho_\alpha(s)$ for all α. In particular, the sheaf axiom (F) is equivalent to the condition tht $\mathcal{G}(U) = \operatorname{Hom}_K(T, \mathcal{F}(U))$ is a sheaf of sets in the ordinary sense, for every $T \in K$.

The axiom (F_0) is stated in the original using this alternate formulation.]

A presheaf on $\mathfrak{B}$ is said to be a *sheaf* on $\mathfrak{B}$ if axiom (F_0) holds.

(3.2.3) A *morphism* $u \colon \mathcal{F} \to \mathcal{G}$ between presheaves on $\mathfrak{B}$ is a natural transformation of functors, *i.e.*, it consists of morphisms $u_V \colon \mathcal{F}(V) \to \mathcal{G}(V)$ for each $V \in \mathfrak{B}$, commuting with the restriction morphisms [Liu, 2.2.10]. The construction of the presheaf $\mathcal{F}'$ on X from the presheaf $\mathcal{F}$ on $\mathfrak{B}$ is functorial.

(3.2.4) If K admits inductive limits, the stalk $\mathcal{F}'_x$ is canonically identified with the inductive limit $\varinjlim_{\mathfrak{B}} \mathcal{F}(V)$, taken over the sets $V \in \mathfrak{B}$ which contain x.

(3.2.5) If $\mathcal{F}$ is a sheaf on X and $\mathcal{F}_{\mathfrak{B}}$ its restriction to $\mathfrak{B}$, then $\mathcal{F} \cong \mathcal{F}'_{\mathfrak{B}}$ canonically. To give a morphism $u \colon \mathcal{F} \to \mathcal{G}$ it suffices to give $u_{\mathfrak{B}} \colon \mathcal{F}_{\mathfrak{B}} \to \mathcal{G}_{\mathfrak{B}}$.

(3.2.6) Given a projective system $(\mathcal{F}_\lambda)$ of sheaves with values in K, the presheaf $\mathcal{F}(U) = \varprojlim \mathcal{F}_\lambda(U)$ is a sheaf, and it is the projective limit of $(\mathcal{F}_\lambda)$ in the category of sheaves with values in K. If K is the category of sets, and $\mathcal{G}_\lambda \subseteq \mathcal{F}_\lambda$ is a system of subsheaves, then $\varprojlim \mathcal{G}_\lambda$ is a subsheaf of $\mathcal{F}$. For sheaves of abelian groups, the functor $\varprojlim$ is left exact.

3.3 Gluing sheaves.

(3.3.1) See Liu, Exercise 2.2.8, for the basic construction. In addition, if $X = \bigcup U_i$ is an open covering, then any sheaf $\mathcal{F}$ is the gluing of its restrictions $\mathcal{F}_i = \mathcal{F}|U_i$ along the identity maps $\mathcal{F}_i|(U_i \cap U_j) = \mathcal{F}_j|(U_i \cap U_j)$.

(3.3.2) A system of morphisms $u_i \colon \mathcal{F}_i \to \mathcal{G}_i$ which commute with the gluing isomoprhisms ϕ_{ij} gives rise to a unique morphism $u \colon \mathcal{F} \to \mathcal{G}$ whose restriction to each U_i is u_i.

(3.3.3) The restriction $\mathcal{F}|V$ is the gluing of the restrictions $\mathcal{F}_i|(V \cap U_i)$.

3.4 Direct images of presheaves.

(3.4.1-2) Given a continuous map $f\colon X \to Y$ and a presheaf $\mathcal{F}$ on X, the *direct image* $f_*\mathcal{F}$ is the presheaf $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ [Liu, p. 37]. If $\mathcal{F}$ is a sheaf, then so is $f_*\mathcal{F}$. The direct image f_* is a functor from presheaves (resp. sheaves) on X with values in K to presheaves (resp. sheaves) on Y with values in K.

(3.4.3) Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, we have $(g \circ f)_*\mathcal{F} = g_*f_*\mathcal{F}$. Given an open set $U \subseteq Y$, and setting $V = f^{-1}(U)$, we have $(f_*\mathcal{F})|U = (f|_V)_*(\mathcal{F}|V)$.

(3.4.4) Assume K admits inductive limits, so stalks make sense, and let $x \in X$, $y = f(x)$. Then there is a canonical morphism $f_x\colon (f_*\mathcal{F})_y \to \mathcal{F}_x$, functorial in $\mathcal{F}$. In general, f_x is *neither injective nor surjective.* Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, let $z = g(y)$; then $(g \circ f)_z = f_x \circ g_y$.

(3.4.5) If f is a homeomorphism of X onto $f(X)$, then f_x is an isomorphism. In particular, this applies to the inclusion $j\colon X \hookrightarrow Y$ of a subspace X of Y.

(3.4.6) If K is the category of groups, rings, etc., and $S \subseteq X$ is the support of $\mathcal{F}$, then the support of $f_*\mathcal{F}$ is contained in the closure $\overline{f(S)}$, but not necessarily in $f(S)$. In particular, if $j\colon X \hookrightarrow Y$ is a closed embedding, then the restriction of $j_*\mathcal{F}$ to $Y \setminus X$ is 0, but it may be non-zero if X is only locally closed.

3.5 Inverse images of presheaves.

(3.5.1) Given $f\colon X \to Y$ and presheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, a morphism $u\colon \mathcal{G} \to f_*\mathcal{F}$ is called an *f-morphism* from $\mathcal{G}$ to $\mathcal{F}$. For all open subsets $U \subseteq X$ and $f(U) \subseteq V \subseteq Y$, u induces morphisms $u_{U,V}\colon \mathcal{G}(V) \to \mathcal{F}(U)$, which commute with restriction to smaller open subsets U', V'. Conversely, any such family of morphisms $u_{U,V}$ commuting with restrictions determines an f-morphism $u\colon \mathcal{G} \to \mathcal{F}$. If K admits all projective limits, and $\mathfrak{B}$, $\mathfrak{B}'$ are bases of the topologies on X and Y, it suffices to give the morphisms $u_{U,V}$ for $U \in \mathfrak{B}$, $V \in \mathfrak{B}'$.

(3.5.2) Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, and f- and g-morphisms $u\colon \mathcal{G} \to f_*\mathcal{F}$, $v\colon \mathcal{H} \to g_*\mathcal{G}$, the composite $w\colon \mathcal{H} \underset{v}{\to} g_*\mathcal{G} \underset{g_*(u)}{\to} g_*f_*\mathcal{F}$ is a $(g \circ f)$-morphism. In this way, one can regard pairs $(X, \mathcal{F})$, where $\mathcal{F}$ is a presheaf on X with values in K, as forming a category, the morphisms $(X, \mathcal{F}) \to (Y, \mathcal{G})$ being pairs (f, u) consisting of a continuous map $f\colon X \to Y$ and an f-morphism $u\colon \mathcal{G} \to \mathcal{F}$.

(3.5.3) Given $f\colon X \to Y$ and a presheaf $\mathcal{G}$ on Y, an *inverse image by f* of $\mathcal{G}$ is a *sheaf* $\mathcal{G}'$ together with an f-morphism $\rho\colon \mathcal{G} \to \mathcal{G}'$ (*i.e.*, a presheaf homomorphism $\mathcal{G} \to f_*\mathcal{G}'$) such that for every *sheaf* $\mathcal{F}$ on X, the map

$$\mathrm{Hom}_X(\mathcal{G}', \mathcal{F}) \to \mathrm{Hom}_f(\mathcal{G}, \mathcal{F}) \underset{\text{def}}{=} \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

induced by composition with ρ is bijective.

Since the pair $(\mathcal{G}', \rho)$ is characterized by a universal property, it is unique up to canonical isomorphism if it exists. Then we denote $\mathcal{G}' = f^*\mathcal{G}$, $\rho = \rho_{\mathcal{G}}$ and call $f^*\mathcal{G}$ *the inverse image sheaf* of $\mathcal{G}$ by f, equipped with the *canonical homomorphism* of presheaves

$$\rho_{\mathcal{G}}\colon \mathcal{G} \to f_*f^*\mathcal{G}.$$

By definition, for any sheaf $\mathcal{F}$ on X, we have a bijective correspondence between homomorphisms of sheaves $v\colon f^*\mathcal{G} \to \mathcal{F}$ on X and homomorphisms of presheaves $u\colon \mathcal{G} \to f_*\mathcal{F}$ on Y, the two being related by $u = f_*(v) \circ \rho_{\mathcal{G}}$.

(3.5.4) Suppose the category K is such that every presheaf $\mathcal{G}$ on Y admits an inverse image $f^*\mathcal{G}$. One can show that this holds under quite general conditions on K; in particular it is true when K is the category of sets, abelian groups, or rings [then $f^*\mathcal{G}$ coincides with the inverse image $f^{-1}\mathcal{G}$ discussed in Liu, p. 37 and Exercises 2.2.6, 2.2.13].

Then f^* is a functor from presheaves on Y to sheaves on X, and the bijective correspondence in (3.5.3) is a functorial isomorphism

$$\mathrm{Hom}_X(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

In other words [although EGA does not put it this way], (f^*, f_*) is a pair of *adjoint functors.* For any sheaf $\mathcal{F}$ on X, there is a canonical homomorphism

$$\sigma_{\mathcal{F}}\colon f^*f_*\mathcal{F} \to \mathcal{F},$$

corresponding to $1 \in \mathrm{Hom}_Y(f_*\mathcal{F}, f_*\mathcal{F})$, such that $u \in \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$ corresponds to $v = \sigma_{\mathcal{F}} \circ f^*(u) \in \mathrm{Hom}_X(f^*\mathcal{G}, \mathcal{F})$.

(3.5.5) Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, suppose that all presheaves on Y and on Z with values in K admit inverse images. Then there is a canonical natural isomorphism of functors $(g \circ f)^* \cong f^* \circ g^*$.

(3.5.6) In the special case $f = 1_X\colon X \to X$, the inverse image $1_X^*\mathcal{F}$ (when it exists) is the *sheaf associated to the presheaf* $\mathcal{F}$ [Liu, 2.2.14]. Then every homomorphism from $\mathcal{F}$ to a *sheaf* $\mathcal{F}'$ factors uniquely through the canonical homomorphism $\mathcal{F} \to 1_X^*\mathcal{F}$.

3.6 Constant and locally constant sheaves. [Liu, 2.2.4, Exercise 2.2.1]

(3.6.1) A *constant presheaf* is a presheaf $\mathcal{F}$ such that the restriction morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is an isomorphism for all U. A *constant sheaf* is the sheaf associated to a constant presheaf. A sheaf $\mathcal{F}$ is *locally constant* if every $x \in X$ has a neighborhood U on which $\mathcal{F}|U$ is constant.

(3.6.2) If X is *irreducible* (this means X is not a union of two proper closed subsets), then the conditions

(a) $\mathcal{F}$ is a constant presheaf
(b) $\mathcal{F}$ is a constant sheaf
(c) $\mathcal{F}$ is a locally constant sheaf

are equivalent.

3.7 Inverse images of presheaves of groups and rings.

(3.7.1) Keep the notation of (3.5.3). When K is the category of sets, one can construct the inverse image $\mathcal{G}' = f^*\mathcal{G}$ [also denoted $f^{-1}\mathcal{G}$, as in Liu] as follows. An element $s' \in \mathcal{G}'(U)$ is a family $(s'_x : x \in U)$, where $s'_x \in \mathcal{G}_{f(x)}$, and for every $x \in X$, the following condition holds: there is a neighborhood V of $f(x)$ in Y, a neighborhood $W \subseteq f^{-1}(V) \cap U$ of x, and a section $s \in \mathcal{G}(V)$ such that s'_z is the germ $s_{f(z)}$ for all $z \in W$. Informally, "s' is given locally by sections of $\mathcal{G}$." The restriction maps are the obvious ones, and the sheaf axiom for $\mathcal{G}'$ holds automatically, by the local nature of the construction.

One proves that $\mathcal{G}'$ as constructed above satisfies the universal property of $f^*\mathcal{G}$. The description of f^* as a functor is immediate: given a morphism $u\colon \mathcal{G}_1 \to \mathcal{G}_2$ and a section $s' = (s'_x) \in \mathcal{G}'_1$, define $f_*(u)(s') = (u_x(s'_x)) \in \mathcal{G}'_2$. When $f = 1_X$ we recover the standard construction of the sheaf associated to a presheaf of sets [Liu, 2.2.15]. The preceding also applies verbatim to presheaves of groups and rings.

(3.7.2) In the setting of (3.7.1), if $\mathcal{G}$ is a *sheaf*, and $\rho\colon \mathcal{G} \to f_*f^*\mathcal{G}$ is the canonical homomorphism, then the induced map on stalks $f_x \circ \rho_{f(x)}$ [see (3.4.4)] is a functorial isomorphism $\mathcal{G}_{f(x)} \to (f^*\mathcal{G})_x$. It follows in particular that $\mathrm{Supp}(f^*\mathcal{G}) = f^{-1}(\mathrm{Supp}(\mathcal{G}))$ and that the inverse image functor f^* on sheaves of abelian groups is *exact*.

3.8 Pseudo-discrete sheaves of topological spaces.

This section is used only in the construction of *formal schemes*, and will be skipped for now.

4.1 Ringed spaces, $\mathcal{A}$-modules and $\mathcal{A}$-algebras.

Note: Liu (2.2.19) considers only *locally* ringed spaces. This extra hypothesis is not necessary, nor is it assumed in EGA. It will hold automatically for schemes and preschemes, and it is irrelevant for the general discussion below.

(4.1.1) A *ringed space* (resp. *topologically ringed space*) $(X, \mathcal{A})$ is a topological space X with a sheaf of rings (resp. topological rings) $\mathcal{A}$. X is called the *underlying space* of $(X, \mathcal{A})$, and $\mathcal{A}$ is its *structure sheaf*, also denoted $\mathcal{O}_X$. One often uses the abbreviation $\mathcal{O}_x$ for the stalk $\mathcal{O}_{X,x}$.

The identity element in the ring of global sections $\mathcal{A}(X)$ is denoted 1.

When we speak of a ringed space $(X, \mathcal{A})$ without indicating otherwise, it will be assumed that $\mathcal{A}$ is a sheaf of *commutative* rings.

Ringed spaces form a category. A morphism $(X, \mathcal{A}) \to (Y, \mathcal{B})$ is a pair (f, ϕ) where $f\colon X \to Y$ is a continuous map and $\phi\colon \mathcal{B} \to \mathcal{A}$ is an f-morphism [see (3.5.1)]. Equivalently, we may specify ϕ by giving the corresponding homomorphism $\phi^\sharp\colon f^*\mathcal{B} \to \mathcal{A}$. [There is a notational conflict here between EGA and Liu, who writes $f^\sharp$ for our ϕ.] Typically one abuses notation and writes f for the pair (f, ϕ).

The composition of $(f, \phi)\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ and $(g, \phi')\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$ is given by $(g \circ f, \phi'')$, where $\phi'' = g_*(\phi) \circ \phi'$ [see (3.5.2)]. This is equivalent to $\phi''^\sharp = \phi^\sharp \circ f^*(\phi'^\sharp)$. Hence if $\phi^\sharp$ and $\phi'^\sharp$ are injective (resp. surjective), then so is $\phi''^\sharp$ (recall from (3.7.2) that f^* is exact). If f is injective and $\phi^\sharp$ is surjective, then (f, ϕ) is a monomorphism in the category of ringed spaces.

(4.1.2) For any subset $M \subseteq X$, we have the ringed space $(M, \mathcal{A}|M)$, called the *restriction* of $(X, \mathcal{A})$ to M. The monomorphism of ringed spaces $(j, \omega)\colon (M, \mathcal{A}|M) \to (X, \mathcal{A})$, where j is the inclusion map and $\omega^\sharp$ is the identity map on $\mathcal{A}|M$, is called the *canonical injection.* The composition of a morphism of ringed spaces $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ with (j, ω) is called the *restriction* of f to M.

(4.1.3) Recall the definition of a *sheaf of $\mathcal{A}$-modules*, or *$\mathcal{A}$-module* for short [Liu, 5.1.1]. If $\mathcal{A}$ is non-commutative, this means a sheaf of left $\mathcal{A}$-modules unless otherwise specified. A sub-$\mathcal{A}$-module of $\mathcal{A}$ is a *sheaf of ideals in* $\mathcal{A}$ (left, right or two-sided).

Assuming $\mathcal{A}$ commutative, and replacing the word "module" by "algebra" in the definition of $\mathcal{A}$-module gives the definition of *$\mathcal{A}$-algebra*. Homomorphisms of $\mathcal{A}$-algebras are defined similarly. One can equivalently define an $\mathcal{A}$-algebra to be an $\mathcal{A}$-module $\mathcal{B}$ equipped with a homomorphism $\mu\colon \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ ("multiplication"), which is associative in the sense that the diagram below commutes.

$$\begin{tikzcd}
\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \arrow[d, "\mu"'] \\
\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \arrow[r, "\mu"] & \mathcal{B}.
\end{tikzcd}$$

The condition that $\mathcal{B}$ is commutative can similarly be expressed by a commutative diagram.

If $\mathcal{M} \subseteq \mathcal{B}$ is a sub-$\mathcal{A}$-module, the sum of the images of the $\mathcal{A}$-module homomorphisms $\bigotimes_{\mathcal{A}}^{n} \mathcal{M} \to \mathcal{B}$ for $n > 0$ is the *sub-$\mathcal{A}$-algebra of $\mathcal{B}$ generated by $\mathcal{M}$.* It is also the sheaf associated to the presheaf which assigns to U the $\mathcal{A}(U)$-subalgebra of $\mathcal{B}(U)$ generated by $\mathcal{M}(U)$.

(4.1.4) A sheaf of rings $\mathcal{A}$ is *reduced* at a point $x \in X$ if the stalk $\mathcal{A}_x$ is reduced (1.1.1); *reduced* if it is reduced at every point. $\mathcal{A}$ is *regular* at x if $\mathcal{A}_x$ is a regular local ring [Liu, 4.27]; *regular* if it is regular at every point. $\mathcal{A}$ is *normal* at x if $\mathcal{A}_x$ is an integrally closed domain [Liu, proof of 4.1.21], *normal* if it is normal at every point. A ringed space $(X, \mathcal{A})$ is said to have any of these properties if $\mathcal{A}$ does.

A sheaf of rings $\mathcal{A}$ is *graded* if it is a direct sum of sheaves of abelian groups $\mathcal{A} = \bigoplus_n \mathcal{A}_n$, satisfying $\mathcal{A}_m \mathcal{A}_n \subseteq \mathcal{A}_{m+n}$. An $\mathcal{A}$-module $\mathcal{M}$ is graded if it is a direct sum of sheaves of abelian groups $\mathcal{M} = \bigoplus_n \mathcal{M}$, satisfying $\mathcal{A}_m \mathcal{M}_n \subseteq \mathcal{M}_{m+n}$. Clearly this makes each stalk $\mathcal{A}_x$ a graded ring, and $\mathcal{M}_x$ a graded module.

(4.1.5) Given a possibly non-commutative ringed space $(X, \mathcal{A})$, one has the bi-functors $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ from sheaves of $\mathcal{A}$-modules (left or right, as appropriate) to sheaves of abelian groups, or more generally, to sheaves of $\mathcal{C}$-modules, where $\mathcal{C}$ is the center of $\mathcal{A}$. [Here are the definitions, for reference. $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{A}(U)} \mathcal{G}(U)$. $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}_{\mathcal{A}|U}(\mathcal{F}|U, \mathcal{G}|U)$; by (3.3.2) this presheaf is a sheaf. See Liu, p. 158 and Exercise 5.1.5(a).]

The stalk $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x$ is canonically isomorphic to $\mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$. There is a canonical homomorphism $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \to \operatorname{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$, which is neither injective nor surjective in general.

The functor $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is right exact in each variable, and commutes with direct limits. The sheaves $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{F}$ and $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{A}$ are canonically isomorphic to $\mathcal{F}$.

The functors $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $\operatorname{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})(X)$ are left exact in each variable (these functors are contravariant in $\mathcal{F}$, so this means they take right exact sequences $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ to left exact sequences). The *dual* of a left $\mathcal{A}$-module $\mathcal{F}$ is the right $\mathcal{A}$-module $\mathcal{F}^{\vee} = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$ [Liu, Exercise 5.1.12].

If $\mathcal{A}$ is commutative, the *p-th exterior power* $\bigwedge^p \mathcal{F}$ of an $\mathcal{A}$-module $\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \bigwedge^p \mathcal{F}(U)$. The canonical homomorphism from this presheaf to its associated sheaf is injective. On stalks, we have $(\bigwedge^p \mathcal{F})_x = \bigwedge^p(\mathcal{F}_x)$. The exterior powers are functorial in $\mathcal{F}$.

(4.1.6) Let $\mathcal{A}$ be a possibly non-commutative sheaf of rings, $\mathcal{I}$ a sheaf of left ideals, $\mathcal{F}$ a left $\mathcal{A}$-module. Then $\mathcal{I}\mathcal{F}$ denotes the submodule of $\mathcal{F}$ which is the image of multiplication $\mathcal{I} \otimes_{\mathbb{Z}} \mathcal{F} \to \mathcal{F}$ (where $\mathbb{Z}$ is the constant sheaf associated to the presheaf $U \mapsto \mathbb{Z}$; if $\mathcal{A}$ is commutative we could also describe $\mathcal{I}$ as the image of $\mathcal{I} \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$). Clearly $(\mathcal{I}\mathcal{F})_x = \mathcal{I}_x\mathcal{F}_x$. It is also immediate that $\mathcal{I}\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{I}(U)\mathcal{F}(U)$, and that if $\mathcal{I}'$ is another sheaf of left ideals, then $\mathcal{I}(\mathcal{I}'\mathcal{F}) = (\mathcal{I}\mathcal{I}')\mathcal{F}$.

(4.1.7) Let $(X_\lambda, \mathcal{A}_\lambda)$ be a family of ringed spaces. For every two indices λ, μ suppose given an open set $V_{\lambda\mu} \subseteq X_\lambda$, and an isomorphism $\phi_{\lambda\mu}\colon (V_{\mu\lambda}, \mathcal{A}_\mu|V_{\mu\lambda}) \tilde{\to} (V_{\lambda\mu}, \mathcal{A}_\lambda|V_{\lambda\mu})$, such that $V_{\lambda\lambda} = X_\lambda$ and $\phi_{\lambda\lambda}$ is the identity. Assume these data satisfy the *gluing condition*: for every three indices λ, μ, ν, if we denote the restriction of $\phi_{\lambda\mu}$ to $V_{\mu\lambda} \cap V_{\mu\nu}$ by $\phi'_{\lambda\mu}$, then $\phi'_{\lambda\mu}$ maps $V_{\mu\lambda} \cap V_{\mu\nu}$ onto $V_{\lambda\mu} \cap V_{\lambda\nu}$, and $\phi'_{\lambda\nu} = \phi'_{\lambda\mu} \circ \phi'_{\mu\nu}$.

Then one can construct a ringed space $(X, \mathcal{A})$ with a covering by open subsets X'_λ such that $(X'_\lambda, \mathcal{A}|X'_\lambda) \cong (X_\lambda, \mathcal{A}_\lambda)$, the sets $V_{\lambda\mu}$ and $V_{\mu\lambda}$ being identified with $X'_\lambda \cap X'_\mu$ so that the given isomorphism $\phi_{\lambda\mu}$ corresponds to the identity. The space $(X, \mathcal{A})$ is said to be constructed by *gluing the spaces* $(X_\lambda, \mathcal{A}_\lambda)$ *along the sets* $V_{\lambda\mu}$ by means of the maps $\phi_{\lambda\mu}$ [Liu, 2.3.33].

4.2 Direct image of an $\mathcal{A}$-module.

(4.2.1) Given a morphism (f, ϕ) of ringed spaces $(X, \mathcal{A}) \to (Y, \mathcal{B})$, and an $\mathcal{A}$-module $\mathcal{F}$, the direct image $f_*\mathcal{F}$ is naturally an $f_*\mathcal{A}$-module, and hence a $\mathcal{B}$-module via $\phi\colon \mathcal{B} \to f_*\mathcal{A}$. This makes f_* a left exact functor from $\mathcal{A}$-modules to $\mathcal{B}$-modules.

(4.2.2) There is a natural transformation of functors

$$f_*(\mathcal{F}) \otimes_{\mathcal{B}} f_*(\mathcal{G}) \to f_*(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}),$$

neither injective nor surjective in general, and a commutative diagram

$$\begin{tikzcd}
f_*(\mathcal{F}) \otimes_{\mathcal{B}} f_*(\mathcal{G}) \otimes_{\mathcal{B}} f_*(\mathcal{H}) \arrow[r] \arrow[d] & f_*(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}) \otimes_{\mathcal{B}} f_*(\mathcal{H}) \arrow[d] \\
f_*(\mathcal{F}) \otimes_{\mathcal{B}} f_*(\mathcal{G} \otimes_{\mathcal{A}} \mathcal{H}) \arrow[r] & f_*(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G} \otimes_{\mathcal{A}} \mathcal{H}).
\end{tikzcd}$$

(4.2.3) Similarly, there is a natural transformation

$$f_* \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{B}}(f_*\mathcal{F}, f_*\mathcal{G}).$$

(4.2.4) If $\mathcal{C}$ is an $\mathcal{A}$-algebra, then $f_*(\mathcal{C})$ is a $\mathcal{B}$-algebra, with multiplication defined by the composition

$$f_*(\mathcal{C}) \otimes_{\mathcal{B}} f_*(\mathcal{C}) \to f_*(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}) \underset{f_*(\mu)}{\to} f_*(\mathcal{C}).$$

Associativity [see (4.1.3)] follows from the commutative diagram in (4.2.2). Similarly, if $\mathcal{F}$ is a $\mathcal{C}$-module, then $f_*(\mathcal{F})$ is naturally an $f_*(\mathcal{C})$-module.

(4.2.5) Consider the special case when $f\colon X \to Y$ is the inclusion of a *closed* subspace. Let $\mathcal{B}' = \mathcal{B}|X = f^*\mathcal{B}$ be the restriction of $\mathcal{B}$ to X. An $\mathcal{A}$-module $\mathcal{M}$ on X can be considered as a $\mathcal{B}'$ module via $\phi^\sharp\colon \mathcal{B}' \to \mathcal{A}$. Then $f_*\mathcal{M}$ is the $\mathcal{B}$-module whose restriction to X is $\mathcal{M}$ and which is 0 outside of X. In this case, the natural transformations in (4.2.2) and (4.2.3) are isomorphisms.

(4.2.6) Given a third space $(Z, \mathcal{C})$ and a morphism $g\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$, the identity $(g \circ f)_* = g_* \circ f_*$ holds as an identity of functors from $\mathcal{A}$-modules to $\mathcal{B}$-modules.

4.3 Inverse image of a $\mathcal{B}$-module.

Notation: if $F = (f, \phi)\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a morphism of ringed spaces, EGA uses f^* to denote the inverse image functor from presheaves on Y to sheaves on X, and F^* to denote the inverse image functor from $\mathcal{B}$-modules to $\mathcal{A}$-modules, which will be discussed below. Since this conflicts with the habit of writing f in place of F, it is customary to write f^{-1} instead of f^* for the topological inverse image functor, as Liu does. The notation f^{-1} will be used below.

(4.3.1) [Liu, 5.1.13] Keep the notation of (4.2.1). The inverse image $f^{-1}(\mathcal{G})$ of a $\mathcal{B}$-module $\mathcal{G}$, constructed as in (3.7.1), is naturally an $f^{-1}(\mathcal{B})$-module. The homomorphism $\phi^\sharp\colon f^{-1}(\mathcal{B}) \to \mathcal{A}$ makes $\mathcal{A}$ an $f^{-1}(\mathcal{B})$-algebra. By extension of scalars, $f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{B})} \mathcal{A}$ is an $\mathcal{A}$-module, called the *inverse image of $\mathcal{G}$ by the morphism* (f, ϕ). We will denote it by $f^*(\mathcal{G})$. Then f^* is a *right exact* functor from $\mathcal{B}$-modules to $\mathcal{A}$-modules. It is not exact in general, since tensoring with $\mathcal{A}$ is only right exact.

One has $f^*(\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{B}_{f(x)}} \mathcal{A}_x$, by (3.7.2) [Liu, 5.1.14].

(4.3.2) f^* commutes with direct limits, and hence with both finite and infinite direct sums.

(4.3.3) f^* commutes with tensor products, in the sense that one has a natural isomorphism

$$f^*(\mathcal{G}_1) \otimes_{\mathcal{A}} f^*(\mathcal{G}_2) \cong f^*(\mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2).$$

(4.3.4) If $\mathcal{C}$ is a $\mathcal{B}$-algebra, then $f^*(\mathcal{C})$ is naturally an $\mathcal{A}$-algebra. In particular, $f^*(\mathcal{B})$ is $\mathcal{A}$ itself. Likewise, if $\mathcal{M}$ is a $\mathcal{C}$-module, then $f^*(\mathcal{M})$ is an $f^*(\mathcal{C})$-module.

(4.3.5) If $\mathcal{I} \subseteq \mathcal{B}$ is a sheaf of ideals, then $f^{-1}(\mathcal{I})$ is a sheaf of ideals in $f^{-1}(\mathcal{B})$, and we have a canonical homorphism $f^*(\mathcal{I}) = f^{-1}(\mathcal{I}) \otimes_{f^{-1}(\mathcal{B})} \mathcal{A} \to \mathcal{A}$, whose image we denote by $f^*(\mathcal{I})\mathcal{A}$, or sometimes simply $\mathcal{I}\mathcal{A}$. Note that $\mathcal{I}\mathcal{A} = \phi^\sharp(f^{-1}(\mathcal{I}))\mathcal{A}$, and hence $(\mathcal{I}\mathcal{A})_x = \phi_x(\mathcal{I}_{f(x)})\mathcal{A}_x$. Given another ideal sheaf $\mathcal{I}' \subseteq \mathcal{B}$, we have $\mathcal{I}(\mathcal{I}'\mathcal{A}) = (\mathcal{I}\mathcal{I}')\mathcal{A}$.

If $\mathcal{F}$ is an $\mathcal{A}$-module, we define $\mathcal{I}\mathcal{F} = (\mathcal{I}\mathcal{A})\mathcal{F}$.

(4.3.6) Given a third space $(Z, \mathcal{C})$ and a morphism $(g, \phi')\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$, we have $(g \circ f)^* = f^* \circ g^*$.

4.4 Relations between direct and inverse images.

(4.4.1-3) [Liu, Exercise 5.1.1] Keep the notation of (4.2.1). There is a canonical isomorphism of functors

$$\mathrm{Hom}_{\mathcal{A}}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{G}, f_*, \mathcal{F}),$$

i.e., (f^*, f_*) is a pair of adjoint functors between $\mathcal{B}$-modules and $\mathcal{A}$-modules. In particular, there are canonical homomorphisms

$$\rho_{\mathcal{G}}\colon \mathcal{G} \to f_*f^*\mathcal{G},$$
$$\sigma_{\mathcal{F}}\colon f^*f_*\mathcal{F} \to \mathcal{F},$$

which determine the above isomorphism as in (3.5.3) and (3.5.4).

More explicitly, if s is a section of $\mathcal{G}$ on an open set $V \subseteq Y$, then $\rho_{\mathcal{G}}(s)$ is the section $s' \otimes 1$ of $f^*\mathcal{G}$ on $f^{-1}(V)$, where s' is given by $s'_x = s_{f(x)}$ for all $x \in f^{-1}(V)$.

Given a homomorphism $u\colon \mathcal{G} \to f_*\mathcal{F}$ and its corresponding homomorphism $u^\sharp\colon f^*\mathcal{G} \to \mathcal{F}$, one has homomorphisms $u_x\colon \mathcal{G}_{f(x)} \to \mathcal{F}_x$ on stalks, defined by composing $(u^\sharp)_x\colon (f^*\mathcal{G})_x \to \mathcal{F}_x$ with the canonical homomorphism $s_x \mapsto s_x \otimes 1$ from $\mathcal{G}_{f(x)}$ to $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{B}_{f(x)}} \mathcal{A}_x$. Equivalently, u_x is the direct limit of the homomorphisms $\mathcal{G}(V) \underset{u}{\to} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}_x$ over open neighborhoods V of $f(x)$.

(4.4.4) Given $u_1\colon \mathcal{G}_1 \to f_*\mathcal{F}_1$, $u_2\colon \mathcal{G}_2 \to f_*\mathcal{F}_2$, denote by $u_1 \otimes u_2$ the homomorphism $u\colon \mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2 \to f_*(\mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2)$ such that $u^\sharp = u_1^\sharp \otimes u_2^\sharp$ [this makes sense by (4.3.3)]. Then u is also the composite

$$\mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2 \overset{u_1 \otimes_{\mathcal{B}} u_2}{\to} (f_*\mathcal{F}_1) \otimes_{\mathcal{B}} (f_*\mathcal{F}_2) \to f_*(\mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2),$$

where the second arrow comes from (4.2.2).

(4.4.5) Let $(\mathcal{G}_\lambda)$ be a direct system of $\mathcal{B}$-modules and $u_\lambda\colon \mathcal{G}_\lambda \to f_*\mathcal{F}$ a system of homomorphisms commuting with the maps in $(\mathcal{G}_\lambda)$. Let $u = \varinjlim u_\lambda$ be the induced morphism from $\mathcal{G} = \varinjlim \mathcal{G}_\lambda$ to $f_*\mathcal{F}$. Then the homomorphisms $u_\lambda^\sharp\colon f^*\mathcal{G}_\lambda \to \mathcal{F}$ commute with the maps in the direct system $(f^*\mathcal{G}_\lambda)$ and we have $u^\sharp = \varinjlim u_\lambda^\sharp$.

(4.4.6) From the defimitions one obtains a natural transformation of functors

$$\gamma\colon \mathcal{H}om_{\mathcal{B}}(\mathcal{G}_1, \mathcal{G}_2) \to f_*(\mathcal{H}om_{\mathcal{A}}(f^*\mathcal{G}_1, f^*\mathcal{G}_2)),$$

and hence a corresponding canonical natural transformation

$$\gamma^\sharp\colon f^*\,\mathcal{H}om_{\mathcal{B}}(\mathcal{G}_1, \mathcal{G}_2) \to \mathcal{H}om_{\mathcal{A}}(f^*\mathcal{G}_1, f^*\mathcal{G}_2).$$

(4.4.7) If $\mathcal{F}$ is an $\mathcal{A}$-algebra, $\mathcal{G}$ is a $\mathcal{B}$-algebra, and $u\colon \mathcal{G} \to f_*\mathcal{F}$ is a $\mathcal{B}$-algebra homomorphism, then $u^\sharp\colon f^*\mathcal{G} \to \mathcal{F}$ is an $\mathcal{A}$-algebra homomorphism, and conversely.

(4.4.8) Given a third ringed space and morphism $(g, \phi')\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$, and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$-modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, with homomorphisms $v\colon \mathcal{G} \to f_*\mathcal{F}$, $v'\colon \mathcal{H} \to g_*\mathcal{G}$, the composite $v'' = g_*(v) \circ v'\colon \mathcal{H} \to (g \circ f)_*\mathcal{F}$ corresponds to $(v'')^\sharp = v^\sharp \circ f^*(v'^\sharp)\colon (g \circ f)^*\mathcal{H} \to \mathcal{F}$.