Synopsis of material from EGA Ch. 0
§2: Irreducible spaces and Noetherian spaces

2.1 Irreducible spaces. [Liu, Section 2.4.2]

(2.1.1) A topological space X is *irreducible* if it is non-empty and not a union of two distinct proper closed subsets. Equivalently: (a) every two non-empty open subsets have non-empty intersection; (b) every non-empty open subset is dense; (c) every open subset is connected.

(2.1.2) A subspace Y of any space X is irreducible iff its closure $\overline{Y}$ is irreducible. In particular the closure of a point $\overline{\{x\}}$ is always irreducible. If $y \in \overline{\{x\}}$ (equivalently $\overline{\{y\}} \subseteq \overline{\{x\}}$) one says that y is a *specialization* of x, or x is a *generalization* of y [Liu, 2.4.10]. A point x such that $X = \overline{\{x\}}$ (if such a point exists) is a *generic point*. A generic point x is contained in every open subset $U \subseteq X$, and is a generic point of U.

(2.1.3) A space is T_0 if for every two distinct points x, y there is an open subset containing one but not the other. A generic point in an irreducible T_0 space is unique. A space X is *quasi-compact* if every open cover of X has a finite sub-cover. Then every non-empty closed subset of X contains a minimal non-empty closed subset M. If X is also T_0, then M consists of a single point (referred to as a *closed point*).

(2.1.4) If X is irreducible, then so is any non-empty open subset of X [Liu, 2.4.5(a)]. If (U_α) is a covering of X by non-empty open subsets, then X is irreducible iff every U_α is irreducible and every $U_\alpha \cap U_\beta$ is non-empty.

(2.1.5) Let X be irreducible and $f\colon X \to Y$ continuous. Then $f(X)$ is irreducible, and if $x \in X$ is a generic point, then $f(x)$ is a generic point in $f(X)$ and also in $\overline{f(X)}$. If Y is irreducible and has a unique generic point y, then $f(X)$ is dense iff $f(x) = y$.

(2.1.6) Every irreducible subspace (and hence also every point) of any space X is contained in some *maximal* irreducible subspace, which is necessarily closed. The maximal irreducible subspaces of X are called its *irreducible components*. A generic point of an irreducible component is not contained in any other irreducible component. If X has only finitely many irreducible components Z_i, and we define $U_j = X \setminus \bigcup_{j \neq i} Z_i$, the sets U_j are open, irreducible, pairwise disjoint, and their union is dense in X.

If $U \subseteq X$ is open, the correspondence $Z \mapsto Z \cap U$ is a bijection from irreducible components of X which meet U, to irreducible components of U [Liu, 2.4.5(b)].

(2.1.7) If X is a finite union of irreducible subspaces Y_i, then the irreducible components of X are the maximal members of the collection of the Y_i's [Liu, 2.4.5(c)]. If $Y \subseteq X$ is a subspace which has finitely many irreducible components, then the closures of its components are the components of $\overline{Y}$.

(2.1.8) Let Y be irreducible with a unique generic point y. Let $f\colon X \to Y$ be continuous. If Z is an irreducible component of X which meets $f^{-1}(y)$, then $f(Z)$ is dense in Y, but not conversely. However, if Z has a generic point, then the converse holds. Moreover, if every irreducible component of X which meets $f^{-1}(y)$ has a generic point, then these components Z are in bijective correspondence with the components $Z \cap f^{-1}(y)$ of $f^{-1}(y)$, with Z and $Z \cap f^{-1}(y)$ having the same generic points.

2.2 Noetherian spaces.

(2.2.1) A space is *Noetherian* if A.C.C. holds for open subsets (equivalently, D.C.C. for closed subsets). It is *locally Noetherian* if every point has a neighborhood which is Noetherian.

(2.2.2) Principle of Noetherian induction: if E is a poset satifying D.C.C., and P is a property of elements of E satisfying the condition "$P(x)$ for all $x < a$ implies $P(a)$," then $P(a)$ holds for all $a \in E$.

(2.2.3) A subspace of a Noetherian space is Noetherian. If X is a finite union of Noetherian subspaces, then X is Noetherian.

(2.2.4) Every Noetherian space is quasi-compact. Conversely, if every open subset of X is quasi-compact, then X is Noetherian.

(2.2.5) A Notherian space has only finitely many irreducible components. This follows easily by Noetherian induction. [Liu, 2.4.9 is a special case.]